

BUNDLES OF SPECTRA AND ALGEBRAIC  $K$ -THEORY

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ABSTRACT. A parametrized spectrum  $E$  is a family of spectra  $E_x$  continuously parametrized by the points  $x \in X$  of a topological space. We take the point of view that a parametrized spectrum is a bundle-theoretic geometric object. When  $R$  is a ring spectrum, we consider parametrized  $R$ -module spectra and show that they give cocycles for the cohomology theory determined by the algebraic  $K$ -theory  $K(R)$  of  $R$  in a manner analogous to the description of topological  $K$ -theory  $K^0(X)$  as the Grothendieck group of vector bundles over  $X$ . We prove a classification theorem for parametrized spectra, showing that parametrized spectra over  $X$  whose fibers are equivalent to a fixed  $R$ -module  $M$  are classified by homotopy classes of maps from  $X$  to the classifying space  $B\mathrm{Aut}_R M$  of the  $A_\infty$  space of  $R$ -module equivalences from  $M$  to  $M$ . In proving the classification theorem for parametrized spectra, we define the notion of a principal  $G$  fibration where  $G$  is an  $A_\infty$  space and prove a similar classification theorem for principal  $G$  fibrations.

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## 1. INTRODUCTION

Contemporary algebraic topology features a vast array of generalized cohomology theories, but with a few exceptions the only theories that have a geometric meaning are the classical examples of ordinary cohomology theories, topological  $K$ -theory and cobordism theories. In this paper we describe the geometry underlying the class of cohomology theories given by the algebraic  $K$ -theory of a ring, or more generally a ring spectrum. The higher algebraic  $K$ -groups  $K_n(R)$  of a ring spectrum  $R$  may be defined as the homotopy groups of the algebraic  $K$ -theory spectrum  $K(R)$ . By the geometry of  $K(R)$ -theory, we do not mean the algebraic geometry of  $\mathrm{spec}(R)$ , but rather a geometric description of the cocycles  $K(R)^*(X)$  for the cohomology theory that the spectrum  $K(R)$  determines. The result is reminiscent of the description of

topological  $K$ -theory  $K^0(X)$  in terms of the Grothendieck group of vector bundles over  $X$ . The analog of vector bundles for  $K(R)$ -theory are parametrized spectra that are modules over the ring spectrum  $R$ . We call these objects  $R$ -bundles. The main result is the following:

**Theorem 1.1.** *Let  $R$  be a connective ring spectrum and let  $K(R)$  be the algebraic  $K$ -theory spectrum of  $R$ . Then for any finite CW complex  $X$ , there is a natural isomorphism*

$$K(R)^0(X) \cong \mathrm{Gr}[\text{lifted finitely free } R\text{-bundles over } X]$$

*between the degree zero cocycles for the  $K(R)$ -theory of  $X$  and the Grothendieck group completion of the abelian monoid of equivalence classes of lifted finitely free  $R$ -bundles over  $X$ .*

We will give a precise meaning to all of the terms occurring in the statement of the theorem in §8 and §9, but for now we note that an  $R$ -bundle  $E$  over  $X$  is finitely free if every fiber  $E_x$  admits a stable equivalence of  $R$ -modules to the  $n$ -fold wedge  $R^{\vee n}$  for some  $n \geq 0$ .

Our geometric description of  $K(R)$ -theory is inspired by previous work. When  $R$  is a discrete ring, Karoubi gave a similar description of the cocycles for  $K(R)$ -theory in terms of fibrations of projective  $R$ -modules [10]. When  $R$  is the connective complex  $K$ -theory spectrum  $ku$ , Baas, Dundas, Richter and Rognes interpreted the cocycles of  $K(ku)$ -theory as 2-vector bundles, which are a categorification of complex vector bundles [5, 6]. In forthcoming work with Jonathan Campbell [7], we explain how to directly compare 2-vector bundles and  $ku$ -bundles.

By definition,  $K(R)^0(X)$  is the group of homotopy classes of maps from  $X$  to the zeroth space of the algebraic  $K$ -theory spectrum, whose homotopy type can be described using Quillen's plus construction:

$$\Omega^\infty K(R) \simeq K_0(R) \times \mathrm{BGL}_\infty^+(R).$$

Here  $K_0(R) = K_0^f(\pi_0 R)$  is the Grothendieck group of free modules over the discrete ring  $\pi_0 R$  and  $\mathrm{BGL}_\infty^+(R)$  is Quillen's plus construction applied to the  $H$ -space  $\mathrm{BGL}_\infty(R) = \mathrm{colim}_n \mathrm{BGL}_n(R)$ , where  $\mathrm{BGL}_n(R)$  is the classifying space of the group-like  $A_\infty$  space  $\mathrm{GL}_n(R) = \mathrm{Aut}_R(R^{\vee n})$  of  $R$ -module equivalences  $R^{\vee n} \rightarrow R^{\vee n}$ . The  $H$ -space structure on  $\mathrm{BGL}_\infty(R)$  arises via the usual block-sum of matrices formula, and we take the plus construction with respect to the commutator subgroup as usual.

One important point is that, unlike the case of vector bundles and complex  $K$ -theory, the plus construction can radically change the homotopy type. This forces the bundles that define cocycles for  $K(R)$ -theory to be *lifted*  $R$ -bundles over  $X$ , meaning  $R$ -bundles defined up to covers of  $X$  with homologically trivial fibers—see §9 for a precise definition.

The term “bundle” is perhaps a little naive: as one continuously varies the basepoint in  $X$ , the fibers of a parametrized spectrum are weakly equivalent, but need not be strictly isomorphic. Put another way, to describe a parametrized spectrum in terms of cocycle data would require a derived or infinitely homotopy coherent descent condition. This point of view naturally leads to the description of parametrized objects in a quasicategory, as developed by Ando, Blumberg, Gepner, Hopkins and Rezk [1–3]. There are many common ideas between their work and ours. Rather than using quasicategories, we follow the foundations of parametrized

stable homotopy theory developed by May and Sigurdsson [18]. In their framework, parametrized spectra are defined in terms of a “total object” over  $X$  instead of cocycle data. Homotopical control of the fiber homotopy type of parametrized spectra is maintained via the framework of Quillen model categories.

Theorem 1.1 follows from a general classification theorem for parametrized  $R$ -module spectra. In this paper, a spectrum means an orthogonal spectrum, and we use the stable model structure on orthogonal ring and module spectra from Mandell-May-Schwede-Shipley [14]. Given an  $R$ -module  $M$ , we say that a parametrized  $R$ -module spectrum  $E$  over  $X$  has fiber  $M$  if the fiber  $E_x$  of  $E$  over every point  $x \in X$  admits a stable equivalence  $E_x \simeq M$  of  $R$ -modules. Let  $\text{Aut}_R M = \text{GL}_1 F^R(M, M)$  be the  $A_\infty$  space of stable equivalences of  $R$ -modules  $M \rightarrow M$ , and let  $B\text{Aut}_R M$  its classifying space.

**Theorem 1.2.** *Let  $X$  be a CW complex, let  $R$  be an  $s$ -cofibrant ring spectrum and let  $M$  be an  $s$ -cofibrant and  $s$ -fibrant  $R$ -module. There is a natural bijection between stable equivalence classes of  $R$ -modules over  $X$  with fiber  $M$  and homotopy classes of maps  $[X, B\text{Aut}_R M]$ .*

Ando-Blumberg-Gepner [2] prove that the quasicategory of functors  $X \rightarrow \mathcal{S}_\infty$  from a Kan complex  $X$  to the quasicategory of spectra  $\mathcal{S}_\infty$  is equivalent to the quasicategory associated to the May-Sigurdsson model category of parametrized spectra over the geometric realization  $|X|$ . Variants of their arguments provide an alternative proof of Theorem 1.2. The proof in this paper is more concrete, using the pullback of a universal bundle to induce the equivalence instead of Lurie’s straightening functor [13, §3.2.1] and the universal properties of certain functors of quasicategories. When  $M = R$ , the theorem says that line  $R$ -bundles over  $X$  are classified by the classifying space  $B\text{GL}_1 R$  of the units of  $R$ . The construction of the line  $R$ -bundle associated to a map  $f: X \rightarrow B\text{GL}_1 R$  is essentially the Thom spectrum of  $f$ ; see Remark 7.7.

In order to prove Theorem 1.2, we develop an associated theory of principal  $A_\infty$  fibrations. Intuitively, a principal  $A_\infty$  fibration is a homotopical version of a principal  $G$ -bundle. Instead of a group,  $G$  is a grouplike  $A_\infty$  space and we let  $G$  act on fibers via weak equivalences instead of isomorphisms. In order to make this notion precise, it is useful to work with a symmetric monoidal product  $\boxtimes$  whose monoids are  $A_\infty$  spaces. This is possible, but at the cost of working within a different category than the category of topological spaces—in fact there are a few different options, which are described and compared in [12] (see also [4, 19]). Since May-Sigurdsson parametrized spectra are built out of orthogonal spectra, it is most natural to use the category of  $\mathcal{I}$ -spaces. An  $\mathcal{I}$ -space is a continuous functor from the category  $\mathcal{I}$  of finite dimensional inner product spaces and isometries to the category of topological spaces. There is a symmetric monoidal product  $\boxtimes$  on  $\mathcal{I}$ -spaces whose monoids we call  $\mathcal{I}$ -FCPs. The category of  $\mathcal{I}$ -FCPs is Quillen equivalent to the category of  $A_\infty$  spaces, which justifies our use of this technology to model  $A_\infty$  multiplications. If  $G$  is an  $\mathcal{I}$ -FCP, then there is a classifying  $\mathcal{I}$ -space  $B^{\boxtimes} G$  built as a two-sided bar construction out of the symmetric monoidal product  $\boxtimes$ . We prove the classification theorem for principal  $G$ -fibrations (Theorem 4.1) using  $B^{\boxtimes} G$ . Under the equivalence of homotopy categories between the homotopy category of  $\mathcal{I}$ -spaces and the homotopy category of spaces,  $B^{\boxtimes} G$  represents the usual classifying space  $BG$  of an  $A_\infty$  space equivalent to  $G$ . The classification

theorem for principal  $A_\infty$  fibrations in the context of  $\mathcal{I}$ -spaces takes the following form.

**Theorem 1.3.** *Let  $G$  be a grouplike  $q$ -cofibrant  $\mathcal{I}$ -FCP and let  $X$  be a CW-complex. Then equivalence classes of principal  $G$ -fibrations over  $X$  are in bijective correspondence with homotopy classes of maps of  $[X, BG]$ .*

§2–§4 are devoted to the proof of this theorem, which has a somewhat classical flavor. While we need model category theory to connect the homotopy theory of  $\mathcal{I}$ -spaces with the homotopy theory of spaces, the classification result itself relies on the homotopy lifting property of Hurewicz fibrations of  $\mathcal{I}$ -spaces. We prove in §5–§8 that the associated bundle construction

$$Y \longmapsto M \wedge_{\Sigma_+^{\infty} \operatorname{Aut}_R M} \Sigma_X^{\infty} Y$$

induces a bijection between equivalence classes of principal  $\operatorname{Aut}_R M$ -fibrations and equivalence classes of  $R$ -bundles with fiber  $M$ , which proves Theorem 1.2 from Theorem 1.3. This material is based entirely on model categories, and can be read independently of the first part of the paper.

*Topological Conventions.* We will rely heavily on the foundations for parametrized homotopy theory developed by May-Sigurdsson [18]. As explained there, it is necessary to leave the category  $\mathcal{U}$  of compactly generated spaces. By a “space” we mean a  $k$ -space, and we denote the category of spaces by  $\mathcal{K}$ . We will use versions of  $\mathcal{I}$ -spaces and  $*$ -modules from [12] based on the category  $\mathcal{K}$  instead of the category  $\mathcal{U}$  of compactly generated spaces. All of the results from that paper are still valid in this context, although some results proven by inducting over cell complexes require the additional assumption of being well-grounded. We will always assume that the base object (denoted by  $B$  or  $X$ ) is compactly generated.

*Outline.* In §2 we set up the general theory of principal fibrations in a symmetric monoidal category. In §3 we introduce the categories of  $\mathcal{I}$ -spaces and  $*$ -modules and prove some homotopical results regarding bar constructions in these categories. §4 is devoted to the classification theorem for principal fibrations of  $\mathcal{I}$ -spaces (Theorem 1.3). In §5 and §6 we switch gears and introduce model category structures on the categories of parametrized  $\mathcal{I}$ -spaces and parametrized spectra. In §7 and §8 we show that principal  $\operatorname{Aut}_R M$ -fibrations are equivalent to  $R$ -bundles with fiber  $M$  and deduce Theorem 1.2 from Theorem 1.3. We complete the proof of Theorem 1.1 in §9. The reader willing to take the classification theorem for principal  $G$ -fibrations for granted can skip directly to the material on parametrized spectra by starting in §5.

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## 2. PRINCIPAL FIBRATIONS

Let  $(\mathcal{C}, \boxtimes, *)$  be a topologically bicomplete closed symmetric monoidal category whose unit  $*$  is the terminal object. Suppose that  $\mathcal{C}$  is equipped with a class of morphisms called weak equivalences that satisfy the two-out-of-three property, and further assume that the associated homotopy category  $\operatorname{Ho} \mathcal{C}$  exists. We will define the notion of a principal fibration structured by a  $\boxtimes$ -monoid in  $\mathcal{C}$ . In the following sections, we will be interested in the cases where  $\mathcal{C}$  is the category of

$\mathbb{I}$ -spaces,  $\mathcal{I}$ -spaces or  $*$ -modules from [12], but for now we work in full generality. In those examples, monoids under  $\boxtimes$  model  $A_\infty$  spaces so we may think of a principal fibration as a principal  $A_\infty$  fibration.

Let  $B$  be an object of  $\mathcal{C}$ . Let  $(X, p) = (p: X \rightarrow B)$  be an object of the category  $\mathcal{C}/B$  of objects of  $\mathcal{C}$  over  $B$ . A point  $b$  of  $B$  is a map  $i_b: * \rightarrow B$  from the terminal object into  $B$ , and we define the fiber of  $(X, p)$  over  $b$  to be the object  $X_b = i_b^* X$  of  $\mathcal{C}$  defined by the following pullback square:

$$\begin{array}{ccc} X_b & \longrightarrow & X \\ \downarrow & & \downarrow p \\ * & \xrightarrow{i_b} & B \end{array}$$

The symmetric monoidal product  $\boxtimes$  on  $\mathcal{C}$  extends to a bifunctor:

$$- \boxtimes -: \mathcal{C} \times \mathcal{C}/B \rightarrow \mathcal{C}/B$$

defined by

$$A \boxtimes (X, p) = (A \boxtimes X \xrightarrow{\pi \boxtimes p} * \boxtimes B \cong B),$$

where  $\pi$  is the map to the terminal object. The category  $\mathcal{C}/B$  is enriched in  $\mathcal{C}$  and the bifunctor  $\boxtimes$  makes  $\mathcal{C}/B$  tensored over  $\mathcal{C}$ .

Taking pullbacks along a map  $f: A \rightarrow B$  in  $\mathcal{C}$  defines a base change functor  $f^*: \mathcal{C}/B \rightarrow \mathcal{C}/A$ . The functor  $f^*$  has a left adjoint  $f_!: \mathcal{C}/A \rightarrow \mathcal{C}/B$  and a right adjoint  $f_*: \mathcal{C}/A \rightarrow \mathcal{C}/B$ . By identifying the category  $\mathcal{C}$  with the category  $\mathcal{C}/*$  of objects over a point, the functor  $X \mapsto X_b$  is the base change functor  $i_b^*: \mathcal{C}/B \rightarrow \mathcal{C}/*$ . Since  $i_b^*$  has both a left and right adjoint, it commutes with limits and colimits. In particular, if  $A$  is an object of  $\mathcal{C}$ , we have a natural isomorphism  $(A \boxtimes X)_b \cong A \boxtimes X_b$ .

Let  $G$  be a monoid under  $\boxtimes$  in  $\mathcal{C}$ . A  $G$ -module over  $B$  is an object  $(E, p)$  of  $\mathcal{C}/B$  with an associative and unital action  $\alpha: G \boxtimes E \rightarrow E$  of  $G$  that is a map over  $B$ . Equivalently,  $E$  is a  $G$ -module and  $p: E \rightarrow B$  is a map of  $G$ -modules, where  $B$  is given the trivial  $G$ -module structure. A map  $E \rightarrow E'$  of  $G$ -modules over  $B$  is a map of  $G$ -modules in the category  $\mathcal{C}/B$ . Each fiber  $E_b$  inherits the structure of a  $G$ -module by applying the functor  $i_b^*$  to the action map  $\alpha$ .

Under our assumptions, the category  $\mathcal{C}$  is tensored over the category  $\mathcal{K}$  of unbased spaces. Let  $I$  denote the unit interval  $[0, 1]$ . Given an object  $X$  of  $\mathcal{C}$ , the tensor  $X \times I$  defines a cylinder object on  $X$ . Using these cylinders, we have an intrinsic notion of homotopy in the category  $\mathcal{C}$ . The category of  $G$ -modules has tensors defined in the underlying category  $\mathcal{C}$ , so we also have a notion of homotopy in the category of  $G$ -modules. We use the terms homotopy equivalence,  $h$ -fibration and  $h$ -cofibration for the usual notions defined in terms of such homotopies. This means that an  $h$ -fibration is a map satisfying the covering homotopy property (CHP) and an  $h$ -cofibration is a map satisfying the homotopy extension property (HEP). As holds in all of our examples, we assume that every homotopy equivalence is a weak equivalence in  $\mathcal{C}$ .

**Definition 2.1.** A principal  $G$ -fibration over  $B$  is a  $G$ -module  $(E, p)$  over  $B$  for which:

- (i) the structure map  $p: E \rightarrow B$  is an  $h$ -fibration in the category of  $G$ -modules, and

(ii) every fiber  $E_b$  admits a chain of weak equivalences of  $G$ -modules to  $G$ .

A map of principal  $G$ -fibrations over  $B$  is a map of  $G$ -modules over  $B$ .

Our first goal is to prove a classification theorem for principal  $G$ -fibrations. Our candidate for a classifying space  $BG$  will be built using the two-sided bar construction based on the symmetric monoidal product  $\boxtimes$  on  $\mathcal{C}$ . Suppose that  $X$  and  $Y$  are left and right  $G$ -modules, respectively. The two-sided bar construction  $B^\boxtimes(X, G, Y)$  is the geometric realization of the simplicial object in  $\mathcal{C}$  whose  $q$ -simplices are the object  $B_q^\boxtimes(X, G, Y) = X \boxtimes G^{\boxtimes q} \boxtimes Y$ . The face maps are defined by the action of  $G$  on  $X$ , the monoid structure of  $G$  and the action of  $G$  on  $Y$ . The degeneracy maps are induced by the unit map  $* \rightarrow G$ . In particular, we may form:

$$B^\boxtimes G = B^\boxtimes(*, G, *) \quad \text{and} \quad E^\boxtimes G = B^\boxtimes(G, G, *).$$

Projecting the copy of  $G$  on the left to the terminal object defines a natural map  $\pi: E^\boxtimes G \rightarrow B^\boxtimes G$  and  $E^\boxtimes G$  is a left  $G$ -module. Furthermore, the usual contracting simplicial homotopy shows that  $E^\boxtimes G$  is contractible.

Consider the equivalence relation on the collection of principal  $G$ -fibrations over  $X$  generated by the maps of principal  $G$ -fibrations. Let  $\mathcal{E}_G(X)$  denote the collection of equivalence classes under this equivalence relation.

**Definition 2.2.** We say that the classification theorem for principal  $G$ -fibrations over  $X$  holds if there is a natural bijection  $\mathcal{E}_G(X) \cong \text{Ho } \mathcal{C}(X, B^\boxtimes G)$  between the set of equivalence classes of principal  $G$ -fibrations and the set of maps  $X \rightarrow B^\boxtimes G$  in the homotopy category of  $\mathcal{C}$ .

We will prove the classification theorem for  $\mathcal{I}$ -spaces in §4. In practice, it is harmless to assume that the class  $\mathcal{E}_G(X)$  is a set, as this is verified in the process of proving the classification theorem.

### 3. HOMOTOPICAL ANALYSIS OF STRUCTURED SPACES

We now introduce the categories of structured spaces that we will work with, referring to [12] for further details. Let  $\mathcal{I}$  denote the category of finite dimensional real inner product spaces and linear isometries  $V \rightarrow W$ . An  $\mathcal{I}$ -space is a continuous functor  $X: \mathcal{I} \rightarrow \mathcal{K}$  from  $\mathcal{I}$  to the category of (unbased) spaces. We write

$$X_{h\mathcal{I}} = \text{hocolim}_{\mathcal{I}} X = \text{hocolim}_{\mathcal{I}^\dagger} X$$

for the homotopy colimit of  $X$ ; here and throughout the paper, a homotopy colimit written over the category  $\mathcal{I}$  is actually taken over a small equivalent subcategory  $\mathcal{I}^\dagger$  that takes into account the Grassmann topology on the space of linear subspaces of a chosen universe  $U$ . The details of this construction are described in §A of [12]. A morphism of  $\mathcal{I}$  spaces  $f: X \rightarrow Y$  is a natural transformation of functors, and we say that  $f$  is a  $q$ -equivalence if the induced map of homotopy colimits  $f_{h\mathcal{I}}: X_{h\mathcal{I}} \rightarrow Y_{h\mathcal{I}}$  is a weak homotopy equivalence of spaces.  $\mathcal{I}\mathcal{K}$  forms a symmetric monoidal category under the bifunctor  $X \boxtimes Y$  defined as the left Kan extension of the external cartesian product

$$\begin{aligned} X \boxtimes Y: \mathcal{I} \times \mathcal{I} &\rightarrow \mathcal{K} \\ (V, W) &\mapsto X(V) \times Y(W) \end{aligned}$$

along the direct sum functor  $\oplus: \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ . We call a monoid under  $\boxtimes$  an  $\mathcal{I}$ -FCP (functor with cartesian product). The category of  $\mathcal{I}$ -FCPs is Quillen equivalent to

the category of algebras over the linear isometries operad (neglecting the symmetric group actions) [12, Theorem 1.3]. Thus an  $\mathcal{I}$ -FCP is a model for an  $A_\infty$  space. Similarly, commutative  $\mathcal{I}$ -FCPs model  $E_\infty$  spaces.

Let  $X$  and  $Y$  be  $\mathcal{I}$ -spaces, and consider the natural transformation

$$l: X_{h\mathcal{I}} \times Y_{h\mathcal{I}} \longrightarrow ((X \boxtimes Y) \circ \oplus)_{h\mathcal{I}^2} \xrightarrow{\oplus_*} (X \boxtimes Y)_{h\mathcal{I}}$$

induced by the canonical map to the left Kan extension  $X \boxtimes Y$  followed by the map of homotopy colimits induced by  $\oplus: \mathcal{I}^2 \rightarrow \mathcal{I}$ . The map  $l$  gives the homotopy colimit functor  $(-)_h\mathcal{I}$  the structure of a lax monoidal functor. In particular, if  $G$  is an  $\mathcal{I}$ -FCP, then the homotopy colimit  $G_{h\mathcal{I}}$  is a topological monoid. We say that  $G$  is grouplike if the topological monoid  $G_{h\mathcal{I}}$  is grouplike, i.e.  $\pi_0 G_{h\mathcal{I}}$  is a group.

We record the following lemma for later use.

**Lemma 3.1.** *The induced map of components*

$$\pi_0 l: \pi_0 X_{h\mathcal{I}} \times \pi_0 Y_{h\mathcal{I}} \longrightarrow \pi_0 (X \boxtimes Y)_{h\mathcal{I}}$$

*is an isomorphism.*

The question of when  $l$  is a  $q$ -equivalence is more subtle, and is best addressed using the notion of flat  $\mathcal{I}$ -spaces. See Sagave-Schlichtkrull [19, §2.24] for a treatment in a slightly different context—in particular their assumption of semistability is not necessary in the present context of the linear isometries diagram category  $\mathcal{I}$ .

*Proof.* There is also a natural transformation in the reverse direction

$$d: (X \boxtimes Y)_{h\mathcal{I}} \xrightarrow{\pi_1 \times \pi_2} (X \times Y)_{h\mathcal{I}} \xrightarrow{\Delta_*} X_{h\mathcal{I}} \times Y_{h\mathcal{I}}$$

given by the product of the projection maps of the form  $\pi_1: X \boxtimes Y \rightarrow X \boxtimes * \cong X$ , followed by the map of homotopy colimits induced by the diagonal functor  $\Delta: \mathcal{I} \rightarrow \mathcal{I}^2$ . A diagram chase shows that  $\pi_1 \circ d \circ l$  is homotopic to the projection onto the first factor (compare with the proof of [19, 2.27]). Combined with the same fact for the other side, this proves that  $d \circ l$  is homotopic to the identity map, and so  $\pi_0 l$  is injective.

For surjectivity, notice that before taking the homotopy colimit, the first map in the composite defining  $l$  is the level-wise quotient map

$$X(V) \times Y(W) \longrightarrow X \boxtimes Y(V \oplus W)$$

to the enriched coend defining the value of the left Kan extension  $X \boxtimes Y$  at  $V \oplus W$ . In particular, after passing to homotopy colimits the map is surjective. The second map  $\oplus_*$  has a section induced by the functor  $\sigma: \mathcal{I} \rightarrow \mathcal{I}^2$  which takes  $V$  to the pair  $(V, 0)$  and similarly for morphisms. Hence the composite  $l$  is also surjective, and thus surjective on  $\pi_0$ .  $\square$

We now turn to  $\mathbb{L}$ -spaces, following [4, 12]. Fix a universe  $U$ , by which we mean an infinite dimensional real inner product space. Let  $\mathbb{L}$  denote the functor on (unbased) spaces defined by  $X \mapsto \mathcal{L}(1) \times X$ , where  $\mathcal{L}(1) = \mathcal{I}_c(U, U)$  is the space of linear isometries from  $U$  to  $U$ . Composition of linear isometries gives  $\mathbb{L}$  the structure of a monad, and we define the category  $\mathbb{L}\mathcal{K}$  of  $\mathbb{L}$ -spaces to be the category of algebras for the monad  $\mathbb{L}$ . A map of  $\mathbb{L}$ -spaces is a  $q$ -equivalence if it is a weak homotopy equivalence of underlying spaces. There is a symmetric monoidal structure  $\boxtimes_{\mathcal{L}}$  on the category of  $\mathbb{L}$ -spaces defined in analogy with the EKMM smash product  $\wedge_{\mathcal{L}}$  of  $\mathcal{L}$ -spectra. The one point space  $*$  is an  $\mathbb{L}$ -space under the trivial action, and for

any  $\mathbb{L}$ -space  $X$  there is a natural unit map  $\lambda: * \boxtimes_{\mathcal{L}} X \longrightarrow X$ . Although  $\lambda$  is not an isomorphism in general, it is always a weak homotopy equivalence. A  $*$ -module is an  $\mathbb{L}$ -space for which the unit map  $\lambda$  is an isomorphism. The full subcategory of  $\mathbb{L}$ -spaces consisting of the  $*$ -modules is a symmetric monoidal category under  $\boxtimes_{\mathcal{L}}$  with unit the one point space  $*$ . The category of (commutative)  $\boxtimes_{\mathcal{L}}$ -monoids is equivalent to the category of  $(E_{\infty}) A_{\infty}$  spaces structured by the linear isometries operad (with or without symmetric group actions taken into account).

Both the category of  $\mathcal{I}$ -spaces and the category of  $*$ -modules satisfy the assumptions placed on the category  $\mathcal{C}$  in §2. Furthermore, they are both well-grounded compactly generated topological monoidal model categories [4, §4.6; 12, 3.4]. In either case, the weak equivalences are the  $q$ -equivalences and we use the terms  $q$ -cofibration and  $q$ -fibration for the cofibrations and fibrations. For both  $\mathcal{I}$ -spaces and  $\mathbb{L}$ -spaces, there are induced model structures on  $\boxtimes$ -monoids and modules over a fixed  $\boxtimes$ -monoid where the weak equivalences and fibrations are created in the underlying category. We will ultimately be interested in  $\mathcal{I}$ -spaces because they are the natural home for the infinite loop space information of orthogonal spectra, and our model for parametrized spectra is based on orthogonal spectra. If instead we used symmetric spectra, then we would employ the category of  $\mathbb{I}$ -spaces, where  $\mathbb{I}$  is Bökstedt's category of finite sets and injections. The results of the next two sections, including the classification theorem, also hold in the category of  $\mathbb{I}$ -spaces.

When  $\mathcal{C}$  is the category of  $*$ -modules, we will write  $B^{\mathcal{L}}(X, G, Y)$  for the bar construction  $B^{\boxtimes}(X, G, Y)$  as a reminder that we are working with the linear isometries operad. When  $\mathcal{C}$  is the category of  $\mathcal{I}$ -spaces, we will write  $B^{\mathcal{I}}(X, G, Y)$  for the bar construction  $B^{\boxtimes}(X, G, Y)$  as a reminder that we are working diagrammatically with diagram category  $\mathcal{I}$ . We say that a monoid  $G$  under  $\boxtimes$  has a non-degenerate basepoint if the unit map  $* \longrightarrow G$  is an  $h$ -cofibration. For both  $\mathcal{I}$ -spaces and  $\mathbb{L}$ -spaces, a  $q$ -cofibrant  $\boxtimes$ -monoid always has a non-degenerate basepoint. We say that a  $\boxtimes$ -monoid  $G$  is grouplike if the monoid  $\pi_0 G$  is a group.

**Lemma 3.2.** *Let  $(\mathcal{C}, \boxtimes, *)$  be a compactly generated topological monoidal model category and let  $G$  be a cofibrant  $\boxtimes$ -monoid.*

- (i) *If  $X$  is a cofibrant right  $G$ -module and  $Y$  is a cofibrant left  $G$ -module, then the two-sided bar construction  $B^{\boxtimes}(X, G, Y)$  is cofibrant.*
- (ii) *If  $G$  has a non-degenerate basepoint, then  $B^{\boxtimes}(-, G, -)$  preserves  $q$ -equivalences in either entry.*

*Proof.* We may assume that  $X, G$  and  $Y$  are cellular objects of the specified types. Since we are working in a monoidal model category, each simplicial level

$$B_q(X, G, Y) = X \boxtimes G^{\boxtimes q} \boxtimes Y$$

is a cellular object of  $\mathcal{C}$ , the degeneracy maps are relative cell complexes, and the face maps are cellular, in the sense that they preserve the cellular filtration. An argument analogous to [8, X.2.7.(i)] then shows that the geometric realization  $B(X, G, Y)$  is also a cellular object of  $\mathcal{C}$ , which proves (i). The second claim follows from the usual glueing lemma argument, since the assumption implies that the simplicial object  $B_*^{\boxtimes}(-, G, -)$  is Reedy cofibrant for the  $h$ -model structure.  $\square$

The technical result at the heart of the classification theorem is establishing that  $E^{\boxtimes} G \longrightarrow B^{\boxtimes} G$  is a quasifibration. We will now do this for  $*$ -modules, following the proof of [3, 5.30]. First we need to recall some facts from [4, §4.5]. Let  $\mathbb{U}$  be

the forgetful functor that takes a  $*$ -module to its underlying space and let  $\mathbb{V}$  be the functor that takes a  $*$ -module  $X$  to the coequalizer  $* \times_{\mathcal{L}(1)} X$  of the action of  $\mathcal{L}(1)$  on a point and on  $X$ . The functor  $\mathbb{U}$  is the right adjoint in the Quillen equivalence of  $*$ -modules with spaces. The functor  $\mathbb{V}$  is symmetric monoidal, so we may use  $\mathbb{V}$  to functorially replace monoids under  $\boxtimes_{\mathcal{L}}$ , i.e.  $A_{\infty}$  spaces, with topological monoids. It is important to note that  $\mathbb{V}$  is not the homotopically meaningful functor from  $*$ -modules to spaces. If this were the case, then every  $E_{\infty}$  space would be weakly equivalent to a commutative topological monoid. However, there is a natural transformation  $\psi: \mathbb{U} \rightarrow \mathbb{V}$  that is a  $q$ -equivalence on  $q$ -cofibrant  $*$ -modules. Note that since  $\mathbb{V}$  is symmetric monoidal and a left adjoint, there is a natural isomorphism relating the  $\boxtimes_{\mathcal{L}}$  bar construction and the usual topological bar construction:

$$\mathbb{V}B^{\mathcal{L}}(Y, G, X) \cong B(\mathbb{V}Y, \mathbb{V}G, \mathbb{V}X).$$

We say that a map  $p: E \rightarrow B$  of  $*$ -modules is a quasifibration if the underlying map of spaces  $\mathbb{U}p$  is a quasifibration.

**Lemma 3.3.** *Let  $G$  be grouplike  $q$ -cofibrant  $\boxtimes_{\mathcal{L}}$ -monoid in  $\mathcal{M}_*$ , let  $X$  be a  $q$ -cofibrant right  $G$ -module and let  $Y$  be a  $q$ -cofibrant left  $G$ -module. Then the projection maps*

$$\pi: B^{\mathcal{L}}(X, G, Y) \rightarrow B^{\mathcal{L}}(X, G, *) \quad \text{and} \quad \pi: B^{\mathcal{L}}(X, G, Y) \rightarrow B^{\mathcal{L}}(*, G, Y)$$

*are quasifibrations.*

*Proof.* The space  $\mathbb{V}G$  is a grouplike topological monoid with non-degenerate base-point, so the projection

$$\pi: B(\mathbb{V}X, \mathbb{V}G, \mathbb{V}Y) \rightarrow B(\mathbb{V}X, \mathbb{V}G, *)$$

is a quasifibration of spaces [15, 7.6]. Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{U}B^{\mathcal{L}}(X, G, Y) & \xrightarrow{\psi} & B(\mathbb{V}X, \mathbb{V}G, \mathbb{V}Y) \\ \mathbb{U}\pi \downarrow & & \downarrow \pi \\ \mathbb{U}B^{\mathcal{L}}(X, G, *) & \xrightarrow{\psi} & B(\mathbb{V}X, \mathbb{V}G, *) \end{array}$$

The bar constructions  $B^{\mathcal{L}}(X, G, Y)$  and  $B^{\mathcal{L}}(X, G, *)$  are  $q$ -cofibrant by Lemma 3.2, and it follows that both instances of  $\psi$  are weak homotopy equivalences. Pick a point  $b \in \mathbb{U}B^{\mathcal{L}}(X, G, *)$  and let  $\psi(b)$  be its image in  $B(*, \mathbb{V}G, Y)$ . Denote the homotopy fibers of  $\mathbb{U}\pi$  and  $\pi$  over  $b$  and  $\psi(b)$  by  $F_b(\mathbb{U}\pi)$  and  $F_{\psi(b)}(\pi)$ , respectively. Since  $\pi$  is a quasifibration, the inclusion of the fiber  $\pi^{-1}(\psi(b)) \rightarrow F_{\psi(b)}(\pi)$  is a weak equivalence. Now consider the following commutative diagram relating fibers and homotopy fibers:

$$\begin{array}{ccc} \mathbb{U}\pi^{-1}(b) & \longrightarrow & \pi^{-1}(\psi(b)) \\ \downarrow & & \downarrow \simeq \\ F_b(\mathbb{U}\pi) & \longrightarrow & F_{\psi(b)}(\pi) \end{array}$$

The map of homotopy fibers is a weak homotopy equivalence by the five lemma. The top horizontal map of fibers may be identified with  $\psi: \mathbb{U}Y \rightarrow \mathbb{V}Y$ , which is a weak equivalence because  $Y$  is  $q$ -cofibrant. Therefore the left vertical map is a

weak homotopy equivalence, which proves that  $\mathbb{U}\pi$  is a quasifibration. The other projection map is proved to be a quasifibration using the same argument.  $\square$

Before proving the analog of this result for  $\mathcal{I}$ -spaces, let us recall some material in [12, §8–§9]. The left adjoint in the Quillen equivalence between  $\mathcal{I}$ -spaces and  $*$ -modules is the functor  $\mathbb{Q}_*: \mathcal{I}\mathcal{U} \rightarrow \mathcal{M}_*$  defined by:

$$X \mapsto * \boxtimes_{\mathcal{I}} (\mathcal{I}(- \otimes U, U) \odot_{\mathcal{I}} X).$$

Here  $\mathcal{I}(- \otimes U, U) \odot_{\mathcal{I}} X$  denotes the tensor product of the functor  $X: \mathcal{I} \rightarrow \mathcal{U}$  and the represented functor  $\mathcal{I}(- \otimes U, U): \mathcal{I}^{\text{op}} \rightarrow \mathbb{L}\mathcal{U}$ , and can be computed using an enriched coend [12, A.4]. The functor  $* \boxtimes_{\mathcal{I}} (-)$  takes  $\mathbb{L}$ -spaces to  $*$ -modules, and is a left adjoint. The composite functor  $\mathbb{Q}_*$  is symmetric monoidal and a left adjoint, so we have natural isomorphisms

$$\mathbb{Q}_* B^{\boxtimes \mathcal{I}}(X, G, Y) \cong B^{\boxtimes \mathcal{I}}(\mathbb{Q}_* X, \mathbb{Q}_* G, \mathbb{Q}_* Y).$$

In particular,  $\mathbb{Q}_* E^{\mathcal{I}} G \cong E^{\mathcal{I}} \mathbb{Q}_* G$  and  $\mathbb{Q}_* B^{\mathcal{I}} G \cong B^{\mathcal{I}} \mathbb{Q}_* G$ . We record the following consistency result, which is a consequence of the weak equivalence  $\psi: \mathbb{U} \rightarrow \mathbb{V}$  on cofibrant objects (see [20] for the analogous result in  $\mathbb{I}$ -spaces).

**Lemma 3.4.** *If  $G$  is a  $q$ -cofibrant  $\mathcal{I}$ -FCP, there is a natural weak homotopy equivalence of spaces  $\mathbb{U}\mathbb{Q}_* B^{\mathcal{I}} G \simeq B(\mathbb{U}\mathbb{Q}_* G)$ . Therefore the  $\mathcal{I}$ -space bar construction  $B^{\mathcal{I}}$  and the usual classifying space functor  $B$  induce the same derived functor on the homotopy category of  $A_{\infty}$  spaces.*

Let  $\mathcal{J}$  denote the subcategory of  $\mathcal{I}$  consisting of subspaces  $V \subset U$  with morphisms the inclusions  $V \rightarrow W$ . There is a natural weak homotopy equivalence of spaces [12, Proposition 9.4]

$$\text{hocolim}_{\mathcal{J}} X \simeq \text{hocolim}_{\mathcal{I}} X,$$

so  $q$ -equivalences of  $\mathcal{I}$ -spaces are detected by taking the homotopy colimit over  $\mathcal{J}$ . Both  $\text{hocolim}_{\mathcal{J}} X$  and  $\text{colim}_{\mathcal{J}} X$  are  $\mathbb{L}$ -spaces, as described in [12, A.12, 9.7]. We now compare these  $\mathbb{L}$ -spaces with  $\mathbb{Q}_* X$ .

**Lemma 3.5.** *Let  $X$  be a  $q$ -cofibrant  $\mathcal{I}$ -space. Then there is a natural chain of  $q$ -equivalences of  $\mathbb{L}$ -spaces:*

$$\text{hocolim}_{\mathcal{J}} X \rightarrow \text{colim}_{\mathcal{J}} X \leftarrow \mathbb{Q}X \leftarrow \mathbb{Q}_* X.$$

*Proof.* The natural projection from the homotopy colimit to the colimit is a weak homotopy equivalence for cofibrant  $\mathcal{I}$ -spaces [12, 9.2]. The second natural transformation depends on a choice of one dimensional subspace of the universe  $U$ , and is also a weak homotopy equivalence [12, 9.7]. The third natural transformation is the weak homotopy equivalence given by the unit map  $\lambda: * \boxtimes_{\mathcal{I}} \mathbb{Q}X \rightarrow \mathbb{Q}X$ .  $\square$

Next we define quasifibrations of  $\mathcal{I}$ -spaces and relate them to quasifibrations of spaces. To do this we will use relative mapping path-spaces. Given a map of spaces  $f: X \rightarrow Y$  and a subspace  $A \subset Y$ , let  $P(f; A) = X \times_Y Y^I \times_Y A$  be the space of paths  $\gamma$  in  $Y$  that start in  $A$ , along with a lift  $x \in X$  of the endpoint:  $f(x) = \gamma(1)$ . Notice that  $P(f; \{b\})$  is the homotopy fiber  $F_b(f)$  of  $f$  above  $b \in Y$ .

Given a map  $\pi: E \rightarrow B$  of  $\mathcal{I}$ -spaces, let  $P\pi$  be the level-wise mapping path space. In other words, the  $\mathcal{I}$ -space  $P\pi$  is given by  $P\pi(V) = P(\pi(V); B(V))$ . The projection  $P\pi \rightarrow B$  is a level-wise  $h$ -fibration of  $\mathcal{I}$ -spaces. We define the homotopy fiber  $F_b(\pi)$  of  $\pi: E \rightarrow B$  over a point  $i_b: * \rightarrow B$  to be the  $\mathcal{I}$ -space given by taking

the level-wise fiber of  $P\pi \rightarrow B$  over  $b$ . We say that  $\pi: E \rightarrow B$  is a quasifibration of  $\mathcal{I}$ -spaces if for every point  $b$  of  $B$ , the inclusion of the fiber into the homotopy fiber  $E_b \rightarrow F_b(\pi)$  is a  $q$ -equivalence of  $\mathcal{I}$ -spaces. Notice that a level-wise quasifibration of  $\mathcal{I}$ -spaces is a quasifibration of  $\mathcal{I}$ -spaces. Unlike the case of topological spaces, an  $h$ -fibration of  $\mathcal{I}$ -spaces need not be a  $q$ -fibration. However, an  $h$ -fibration of  $\mathcal{I}$ -spaces is still a quasifibration of  $\mathcal{I}$ -spaces.

**Lemma 3.6.** *Let  $\pi: E \rightarrow B$  be a map of  $\mathcal{I}$ -spaces. For every point  $i_b: * \rightarrow B$  in the base, the homotopy fiber  $F_b(\pi)$  of  $\pi$  fits into a long exact sequence of homotopy groups*

$$\cdots \rightarrow \pi_i F_b(\pi) \rightarrow \pi_i E \rightarrow \pi_i B \rightarrow \pi_{i-1} F_b(\pi) \rightarrow \cdots$$

that ends with:

$$\cdots \rightarrow \pi_0 F_b(\pi) \rightarrow \pi_0 E \rightarrow \pi_0 B.$$

Assume further that  $E$ ,  $B$  and every fiber  $E_b$  are  $q$ -cofibrant  $\mathcal{I}$ -spaces. If the map  $\mathbb{Q}_*\pi: \mathbb{Q}_*E \rightarrow \mathbb{Q}_*B$  is a quasifibration of spaces, then the map  $\pi$  is a quasifibration of  $\mathcal{I}$ -spaces.

*Proof.* We will prove the second claim first. Let  $P\pi$  be the level-wise mapping path space of  $\pi: E \rightarrow B$ . Taking homotopy colimits, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \text{hocolim}_{\mathcal{J}} E_b & \xrightarrow{\cong} & \text{colim}_{\mathcal{J}} E_b & \xleftarrow{\cong} & \mathbb{Q}_* E_b \\
 \swarrow & & \downarrow & & \downarrow \\
 \text{hocolim}_{\mathcal{J}} F_b(\pi) & \xrightarrow{\quad} & \text{hocolim}_{\mathcal{J}} P\pi & \xleftarrow{\cong} & \text{hocolim}_{\mathcal{J}} E & \xrightarrow{\cong} & \text{colim}_{\mathcal{J}} E & \xleftarrow{\cong} & \mathbb{Q}_* E \\
 \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 & & \downarrow \pi_4 & & \downarrow \mathbb{Q}_*\pi \\
 B\mathcal{J} & \xrightarrow{i_b} & \text{hocolim}_{\mathcal{J}} B & \xlongequal{\quad} & \text{hocolim}_{\mathcal{J}} B & \xrightarrow{\cong} & \text{colim}_{\mathcal{J}} B & \xleftarrow{\cong} & \mathbb{Q}_* B
 \end{array}$$

The horizontal maps in the right three columns are components of the chain of natural transformations in Lemma 3.5. They are  $q$ -equivalences by the cofibrancy assumptions. The other displayed equivalence is induced by the level-wise homotopy equivalence  $E \rightarrow P\pi$ .

If  $B\mathcal{J}$  were a point, we could work directly with the homotopy fibers of  $\pi_2$  and  $\pi_3$ . However,  $B\mathcal{J}$  is only contractible, so we must work with the relative mapping path spaces  $P(\pi_i; B\mathcal{J})$ . For  $\pi_2$  and  $\pi_3$ , we identify  $B\mathcal{J}$  with its image in  $\text{hocolim}_{\mathcal{J}} B$  under the map of homotopy colimits induced by the inclusion  $i_b: * \rightarrow B$ . Since the functor  $\mathbb{Q}_*$  is strong symmetric monoidal, the map  $\mathbb{Q}_* i_b: \mathbb{Q}_*(*) \rightarrow \mathbb{Q}_* B$  is the inclusion of a point, whose image we denote by  $b \in \mathbb{Q}_* B$ . The relative mapping path spaces of  $\mathbb{Q}_*\pi$  and  $\pi_4$  over  $b$  and its image  $b \in \text{colim}_{\mathcal{J}} B$  are the homotopy fibers  $F_b(\mathbb{Q}_*\pi)$  and  $F_b(\pi_4)$ . This notation allows us to make the identifications of fibers  $(\mathbb{Q}_* E)_b = \mathbb{Q}_* E_b$  and  $(\text{colim}_{\mathcal{J}} E)_b = \text{colim}_{\mathcal{J}} E_b$ . We now have a commutative

diagram of fibers and relative mapping path spaces:

(3.1)

$$\begin{array}{ccccccc}
 \operatorname{hocolim}_{\mathcal{J}} F_b(\pi) & \longleftarrow & \operatorname{hocolim}_{\mathcal{J}} E_b & \xrightarrow{\simeq} & \operatorname{colim}_{\mathcal{J}} E_b & \xleftarrow{\simeq} & \mathbb{Q}_* E_b \\
 \downarrow \simeq & \swarrow & \searrow & & \downarrow & & \downarrow \\
 P(\pi_1; B\mathcal{J}) & \xrightarrow{\simeq} & P(\pi_2; B\mathcal{J}) & \xleftarrow{\simeq} & P(\pi_3; B\mathcal{J}) & \xrightarrow{\simeq} & F_b(\pi_4) \xleftarrow{\simeq} F_b(\mathbb{Q}_*\pi)
 \end{array}$$

The two right vertical maps are the canonical inclusions of the fiber into the homotopy fiber. The maps along the bottom row arise by the functoriality of the relative mapping path-space construction, and the right three are weak homotopy equivalences because they are induced by weak homotopy equivalences of total and base spaces. The left vertical map  $\operatorname{hocolim}_{\mathcal{J}} F_b\pi \rightarrow P(\pi_1, B\mathcal{J})$  is a weak homotopy equivalence because  $B\mathcal{J}$  is contractible.

To prove that the map  $P(\pi_1; B\mathcal{J}) \rightarrow P(\pi_2; B\mathcal{J})$  is a weak homotopy equivalence, we compare it to the map of the homotopy fibers of  $\pi_1$  and  $\pi_2$ :

$$\begin{array}{ccc}
 P(\pi_1; *) & \longrightarrow & P(\pi_2; *) \\
 \downarrow & & \downarrow \\
 P(\pi_1; B\mathcal{J}) & \longrightarrow & P(\pi_2; B\mathcal{J})
 \end{array}$$

Here  $*$   $\in B\mathcal{J}$  is any choice of basepoint and the vertical maps are induced by its inclusion into  $B\mathcal{J}$ . The map  $P(\pi_1; *) \rightarrow P(\pi_2; *)$  of homotopy fibers is a weak homotopy equivalence by [12, Lemma 15.7.(i)]. Since  $B\mathcal{J}$  is contractible, the vertical maps are also weak homotopy equivalences. Thus the map  $P(\pi_1; B\mathcal{J}) \rightarrow P(\pi_2; B\mathcal{J})$  in diagram (3.1) is a weak homotopy equivalence.

We now complete the proof of the second claim in the lemma. If  $\mathbb{Q}_*\pi$  is a quasifibration of spaces, then the right vertical map  $\mathbb{Q}_*(E_b) \rightarrow F_b(\mathbb{Q}_*\pi)$  in diagram (3.1) is a weak equivalence. It then follows from the diagram that  $E_b \rightarrow F_b(\pi)$  is a  $q$ -equivalence and so  $\pi$  is a quasifibration of  $\mathcal{I}$ -spaces.

For the first claim, notice that diagram (3.1) contains a natural chain of weak homotopy equivalences (that do not depend on the cofibrancy assumptions)

$$\operatorname{hocolim}_{\mathcal{J}} F_b(\pi) \simeq P(\pi_3; B\mathcal{J}).$$

Since  $B\mathcal{J}$  is contractible, this implies that  $\operatorname{hocolim}_{\mathcal{J}} F_b(\pi)$  is weak homotopy equivalent to the homotopy fiber of  $\pi_3$ :  $\operatorname{hocolim}_{\mathcal{J}} E \rightarrow \operatorname{hocolim}_{\mathcal{J}} B$ , from which we deduce the long exact sequence of homotopy groups.  $\square$

Before proving that the projection  $E^{\mathcal{I}}G \rightarrow B^{\mathcal{I}}G$  is a quasifibration, we need to be able to identify its fiber.

**Lemma 3.8.** *The fiber  $F$  of the projection  $\pi_X: X \boxtimes_{\mathcal{I}} Y \rightarrow X$  over a point  $i_x: * \rightarrow X$  is naturally isomorphic to  $Y$ .*

*Proof.* Define a map  $f: X \boxtimes_{\mathcal{I}} Y \rightarrow F$  using the universal mapping property of the pullback:

$$\begin{array}{ccccc}
 X \boxtimes_{\mathcal{I}} Y & \xrightarrow{\pi_Y} & Y & & \\
 \searrow f & & \searrow i_x \boxtimes \text{id} & & \\
 & F & \xrightarrow{i_F} & X \boxtimes_{\mathcal{I}} Y & \\
 & \downarrow & & \downarrow \pi_X & \\
 & * & \xrightarrow{i_x} & X & 
 \end{array}$$

A diagram chase shows that the composite

$$Y \xrightarrow{i_x \boxtimes \text{id}} X \boxtimes_{\mathcal{I}} Y \xrightarrow{f} F \xrightarrow{i_F} X \boxtimes_{\mathcal{I}} Y \xrightarrow{\pi_Y} Y$$

is the identity map of  $Y$ . Thus  $Y$  is a retract of  $F$ , and it suffices to show that  $\pi_Y \circ i_F$  is a monomorphism. We will use the coequalizer description of  $X \boxtimes Y$  from [12, 18.6] to show that this is true after evaluation at  $\mathbf{R}^m$ . The map  $\pi_Y$  is induced by the map of coequalizer diagrams with domain

$$\begin{array}{c}
 \coprod_{a+b+c=m} O(m) \times_{O(a) \times O(b) \times O(c)} X(\mathbf{R}^a) \times Y(\mathbf{R}^c) \\
 \Downarrow \\
 \coprod_{a+b=m} O(m) \times_{O(a) \times O(b)} X(\mathbf{R}^a) \times Y(\mathbf{R}^b) \\
 \downarrow \\
 X \boxtimes_{\mathcal{I}} Y(\mathbf{R}^m).
 \end{array}$$

that collapses the entries  $X(-)$  to a point. Let us write  $x_n \in X(\mathbf{R}^n)$  for the image of  $i_x: * \rightarrow X$  at level  $\mathbf{R}^n$ . Let  $(\varphi, x_a, y)$  and  $(\varphi', x_{a'}, y')$  be two points of the fiber  $F$  coming from the  $(a, b)$  and  $(a', b')$  summands, respectively, and suppose that they have the same image under  $\pi_Y$ . Then  $(\varphi|_{\mathbf{R}^b})(y) = (\varphi'|_{\mathbf{R}^{b'}})(y')$  and it follows that  $(\varphi, x_a, y)$  and  $(\varphi', x_{a'}, y')$  are identified under the coequalizer. Therefore  $(\pi_Y \circ i_F)(V)$  is a monomorphism.  $\square$

**Lemma 3.9.** *Suppose that  $G$  is an  $\mathcal{I}$ -FCP,  $X$  is a right  $G$ -module and  $Y$  is a left  $G$ -module. Consider the projection map of bar constructions*

$$\pi: B^{\mathcal{I}}(X, G, Y) \rightarrow B^{\mathcal{I}}(X, G, *)$$

*induced by the unique map  $Y \rightarrow *$ . The fiber of  $\pi$  over any point  $* \rightarrow B^{\mathcal{I}}(X, G, *)$  is naturally isomorphic to  $Y$ .*

*Proof.* By the previous lemma, the fiber over a point  $* \rightarrow B_q^{\mathcal{I}}(X, G, *)$  in simplicial degree  $q$  can be canonically identified with  $Y$ . Since every point in the geometric realization has a unique representation as a non-degenerate point in some simplicial degree, the claim follows.  $\square$

**Proposition 3.10.** *Let  $G$  be a grouplike  $q$ -cofibrant  $\mathcal{I}$ -FCP. Let  $X$  be a right  $G$ -module and let  $Y$  be a left  $G$ -module. Then the projection maps*

$$\pi: B^{\mathcal{I}}(X, G, Y) \rightarrow B^{\mathcal{I}}(X, G, *) \quad \text{and} \quad \pi: B^{\mathcal{I}}(X, G, Y) \rightarrow B^{\mathcal{I}}(*, G, Y)$$

are quasifibrations of  $\mathcal{I}$ -spaces.

*Proof.* We will do the first map as the proof for the other works in the same way. Let us first prove the claim under the additional assumption that  $X$  and  $Y$  are  $q$ -cofibrant  $G$ -modules. Let  $b: * \rightarrow B^{\mathcal{I}}(X, G, *)$  be a point in the base. By Lemma 3.9, the fiber of  $\pi$  over  $b$  is isomorphic to  $Y$ , and in particular is  $q$ -cofibrant. The  $\mathcal{I}$ -spaces  $B^{\mathcal{I}}(X, G, Y)$  and  $B^{\mathcal{I}}(X, G, *)$  are also  $q$ -cofibrant by Lemma 3.2.(i). The cofibrancy hypotheses in Lemma 3.6 are now satisfied, so it remains to show that  $\mathbb{Q}_*\pi$  is a quasifibration of spaces. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}_*B^{\mathcal{I}}(X, G, Y) & \xrightarrow{\mathbb{Q}_*\pi} & \mathbb{Q}_*B^{\mathcal{I}}(X, G, *) \\ \downarrow \cong & & \downarrow \cong \\ B^{\mathcal{L}}(\mathbb{Q}_*X, \mathbb{Q}_*G, \mathbb{Q}_*Y) & \xrightarrow{\pi} & B^{\mathcal{L}}(\mathbb{Q}_*X, \mathbb{Q}_*G, *) \end{array}$$

The vertical maps are the canonical isomorphisms interchanging the functor  $\mathbb{Q}_*$  and the bar construction. Since the functor  $\mathbb{Q}_*$  is left Quillen,  $\mathbb{Q}_*G$  is a grouplike  $q$ -cofibrant monoid in  $*$ -modules with non-degenerate basepoint. Thus Lemma 3.3 applies, proving that the bottom horizontal arrow is a quasifibration of spaces.

To deduce the general case, notice that cofibrant approximations of  $X$  and  $Y$  induce  $q$ -equivalences of the base and total objects by Lemma 3.2.(ii), as well as  $q$ -equivalences of fibers and homotopy fibers.  $\square$

#### 4. THE CLASSIFICATION OF PRINCIPAL FIBRATIONS

Let  $G$  be a grouplike  $q$ -cofibrant  $\mathcal{I}$ -FCP. This section is devoted to the proof of the classification theorem for principal  $G$ -fibrations. We will only work with  $\mathcal{I}$ -spaces, so let us drop the superscript  $\mathcal{I}$  from the notation for bar constructions. Attentive readers will notice that this section follows the method of proof of the classification theorem for fibrations of spaces from [15]. There is some subtlety in our more general context because not all  $\mathcal{I}$ -spaces are fibrant.

Let  $[X, Y]$  denote the set of homotopy equivalence classes of maps of  $\mathcal{I}$ -spaces  $X \rightarrow Y$ , defined in terms of the tensor  $X \times I$ . When  $X$  is  $q$ -cofibrant,  $X \times I$  is a cylinder object in the  $q$ -model structure on  $\mathcal{I}$ -spaces. Thus for  $X$   $q$ -cofibrant and  $Y$   $q$ -fibrant, there is a natural isomorphism  $[X, Y] \cong \mathrm{Ho} \mathcal{I}\mathcal{K}(X, Y)$  between homotopy classes of maps and maps in the homotopy category of  $\mathcal{I}$ -spaces. The classification theorem for principal  $G$ -fibrations, in the sense of definition 2.2, follows from the following theorem.

**Theorem 4.1.** *Let  $X$  be a  $q$ -cofibrant  $\mathcal{I}$ -space and let  $B'G$  be a  $q$ -fibrant approximation of  $B^{\mathcal{I}}G = BG$ . There is a natural bijection  $\mathcal{E}_G(X) \cong [X, B'G]$  between equivalence classes of principal  $G$ -fibrations over  $X$  and homotopy classes of maps  $X \rightarrow B'G$ .*

A CW complex  $X$  determines a constant  $\mathcal{I}$ -space  $\Delta X(V) = X$ , which is  $q$ -cofibrant by the description of the generating  $q$ -cofibrations in [12, §15]. By the consistency result in Lemma 3.4, the equivalence of homotopy categories  $\mathrm{Ho} \mathcal{I}\mathcal{K} \simeq \mathrm{Ho} \mathcal{K}$  gives a natural bijection

$$[X, B\mathbb{Q}_*G] \cong \mathrm{Ho} \mathcal{K}(X, B\mathbb{Q}_*G) \cong \mathrm{Ho} \mathcal{I}\mathcal{K}(\Delta X, B^{\boxtimes}G).$$

We thus deduce Theorem 1.3 from Theorem 4.1.

The proof of the Theorem 4.1 takes up the rest of this section. We will need the following version of Whitehead's theorem. The proof is standard model category theory.

**Lemma 4.2.** *Suppose that  $X$  is a  $q$ -cofibrant  $\mathcal{I}$ -space.*

(i) *A  $q$ -equivalence  $f: A \rightarrow B$  of  $q$ -fibrant  $\mathcal{I}$ -spaces induces an isomorphism*

$$f_*: [X, A] \rightarrow [X, B].$$

(ii) *If  $p: B \rightarrow X$  is a  $q$ -equivalence of  $q$ -fibrant  $\mathcal{I}$ -spaces, then there exists a map  $s: X \rightarrow B$  such that  $p \circ s$  is homotopic to  $\text{id}_X$ . Up to homotopy,  $s$  is unique among maps with this property. Furthermore, the construction of  $s$  is natural up to homotopy.*

It is straightforward to verify that principal  $G$ -fibrations behave as expected under pullback:

**Proposition 4.3.** *Let  $p: E \rightarrow B$  be a principal  $G$ -fibration and suppose that  $f: A \rightarrow B$  is a map of  $\mathcal{I}$ -spaces. Then the pullback  $(f^*E, f^*p)$  of  $(E, p)$  along  $f$  is a principal  $G$ -fibration. Furthermore, if  $f, g: A \rightarrow B$  are homotopic maps of  $\mathcal{I}$ -spaces, then there is a homotopy equivalence of principal  $G$ -fibrations  $f^*E \simeq g^*E$ .*

We will use a version of the approximation functor  $\Gamma$  of [15], adapted to the setting of  $\mathcal{I}$ -spaces. The purpose of  $\Gamma$  is to replace quasifibrations with  $h$ -fibrations that still have the correct fiber homotopy type. Given an  $\mathcal{I}$ -space  $B$ , let  $B^{[0, \infty]}$  be the cotensor of the  $\mathcal{I}$ -space  $B$  with the space  $[0, \infty]$ , and let  $\underline{\Pi}B = B^{[0, \infty]} \times [0, \infty)$  be the tensor with  $[0, \infty)$ . We think of  $\underline{\Pi}B$  as the space of all paths in  $B$  with a specified time for endpoint evaluation. The inclusion  $[0, \infty) \rightarrow [0, \infty]$  and the counit of the tensor/cotensor adjunction define the endpoint evaluation map

$$e: \underline{\Pi}B \rightarrow B^{[0, \infty]} \times [0, \infty] \xrightarrow{\epsilon} B.$$

To restrict to paths that are constant after a compact amount of time, let  $+$  be the shift of endpoint map

$$\begin{aligned} +: \underline{\Pi}B \times (0, \infty) &\rightarrow \underline{\Pi}B \\ (\gamma, s, \epsilon) &\mapsto (\gamma, s + \epsilon), \end{aligned}$$

and define  $\Pi B$  to be the equalizer of the following two composites:

$$\Pi B = \text{equalizer} \begin{cases} \underline{\Pi}B \xrightarrow{\eta} (\underline{\Pi}B \times (0, \infty))^{(0, \infty)} \xrightarrow{+(0, \infty)} \underline{\Pi}B^{(0, \infty)} \xrightarrow{e^{(0, \infty)}} B^{(0, \infty)} \\ \underline{\Pi}B \xrightarrow{e} B \xrightarrow{\text{const}} B^{(0, \infty)} \end{cases}$$

The  $\mathcal{I}$ -space  $\Pi B$  is the appropriate version of Moore paths on an  $\mathcal{I}$ -space. Given an  $\mathcal{I}$ -space  $(E, p)$  over  $B$ , define the  $\mathcal{I}$ -space  $\Gamma E$  as the pullback of  $p: E \rightarrow B$  along the evaluation at 0 map  $e_0: \Pi B \rightarrow B$ . Let  $\Gamma p$  be the induced endpoint evaluation map:

$$\Gamma p: \Gamma E \rightarrow \Pi B \xrightarrow{e} B.$$

Then  $(\Gamma E, \Gamma p)$  is an  $\mathcal{I}$ -space over  $B$ , and  $\Gamma p$  is an  $h$ -fibration of  $\mathcal{I}$ -spaces. We record the basic properties of  $\Gamma$ .

**Proposition 4.4.** *The construction  $\Gamma$  is functorial and satisfies the following properties.*

- (i) If  $(E, p)$  is a  $G$ -module over  $B$  and every homotopy fiber of  $(E, p)$  admits a chain of  $q$ -equivalences of  $G$ -modules  $F_b(p) \simeq G$ , then  $(\Gamma E, \Gamma p)$  is a principal  $G$ -fibration over  $B$ . In particular, if  $p$  is a quasifibration of  $\mathcal{I}$ -spaces and every fiber  $E_b$  is  $q$ -equivalent to  $G$ , then  $(\Gamma E, \Gamma p)$  is a principal  $G$ -fibration over  $B$ .
- (ii) Suppose that  $(D, p) \rightarrow (E, q)$  is a  $q$ -equivalence of  $G$ -modules over  $B$ . Then the induced map  $(\Gamma D, \Gamma p) \rightarrow (\Gamma E, \Gamma q)$  is a  $q$ -equivalence of principal  $G$ -fibrations.
- (iii) The map  $\eta: (E, p) \rightarrow (\Gamma E, \Gamma p)$  defined by the inclusion into constant paths is a homotopy equivalence of  $G$ -modules over  $B$ . If  $p$  is a quasifibration of  $\mathcal{I}$ -spaces, then  $\eta$  restricts to a  $q$ -equivalence on fibers.

**Lemma 4.5.** Suppose that  $Y$  is a left  $G$ -module that admits a  $q$ -equivalence of  $G$ -modules  $Y \simeq G$  and let  $y: * \rightarrow G^{\boxtimes q} \boxtimes Y$  be a point. Then the composite

$$G \cong G \boxtimes * \xrightarrow{\text{id} \boxtimes y} G \boxtimes (G^{\boxtimes q} \boxtimes Y) \xrightarrow{\alpha} Y$$

of  $y$  with the  $(q+1)$ -fold iteration of the left  $G$ -module structure map  $\alpha$  is a  $q$ -equivalence.

*Proof.* Choose a zig-zag of  $q$ -equivalences of  $G$ -modules between  $G$  and  $Y$ . In the following diagram, the upper right square represents the induced chain of commuting naturality squares relating the induced actions of the topological monoid  $G_{h\mathcal{I}}$  on  $G_{h\mathcal{I}}$  and  $Y_{h\mathcal{I}}$ . The bottom portion of the diagram is induced by the lax monoidal structure map  $X_{h\mathcal{I}} \times Y_{h\mathcal{I}} \rightarrow (X \boxtimes Y)_{h\mathcal{I}}$  of the homotopy colimit functor, and the lower right square commutes by definition.

$$\begin{array}{ccccc}
 G_{h\mathcal{I}} \times * & \xrightarrow{\text{id} \times g} & G_{h\mathcal{I}} \times G_{h\mathcal{I}}^q \times G_{h\mathcal{I}} & \xrightarrow{\mu} & G_{h\mathcal{I}} \\
 \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\
 G_{h\mathcal{I}} \times B\mathcal{I} & \xrightarrow{\text{id} \times y} & G_{h\mathcal{I}} \times G_{h\mathcal{I}}^q \times Y_{h\mathcal{I}} & \xrightarrow{\alpha} & Y_{h\mathcal{I}} \\
 \downarrow & & \downarrow & & \parallel \\
 (G \boxtimes *)_{h\mathcal{I}} & \xrightarrow{\text{id} \boxtimes y} & (G \boxtimes G^{\boxtimes q} \boxtimes Y)_{h\mathcal{I}} & \xrightarrow{\alpha} & Y_{h\mathcal{I}}
 \end{array}$$

Since  $B\mathcal{I}$  is contractible and  $\pi_0 Y_{h\mathcal{I}} \cong \pi_0 G_{h\mathcal{I}}$ , we may choose a point  $g \in G_{h\mathcal{I}}^q \times G_{h\mathcal{I}}$  so that the upper left hand square represents a chain of commuting squares. The vertical maps on the left are both homotopy equivalences, and the top composite is a homotopy equivalence because  $G_{h\mathcal{I}}$  is a grouplike topological monoid. Therefore the bottom composite is a weak homotopy equivalence.  $\square$

Suppose that  $X$  is a right  $G$ -module,  $Y$  is a left  $G$ -module, and that we are given a map of  $\mathcal{I}$ -spaces  $f: X \boxtimes_G Y \rightarrow Z$ . We will write  $\epsilon(f)$  for the composite

$$\epsilon(f): B(X, G, Y) \rightarrow X \boxtimes_G Y \xrightarrow{f} Z$$

of  $f$  with the canonical projection from the two sided bar construction. In particular, the canonical isomorphism  $\alpha: G \boxtimes_G Y \rightarrow Y$  induces a map of left  $G$ -modules  $\epsilon(\alpha): B(G, G, Y) \rightarrow Y$ , and the appropriate version of a standard simplicial homotopy argument [17, 9.8] shows that  $\epsilon(\alpha)$  is a homotopy equivalence.

Now suppose that  $p: Y \rightarrow X$  is a principal  $G$ -fibration. Since the action of  $G$  on  $X$  is trivial, the map  $p$  factors through the quotient of  $Y$  by the action of  $G$  to give a map  $\bar{p}: * \boxtimes_G Y \rightarrow X$ .

**Proposition 4.6.** *The map*

$$\epsilon(\bar{p}): B(*, G, Y) \rightarrow X$$

*is a  $q$ -equivalence.*

*Proof.* Consider the following commutative diagram.

$$\begin{array}{ccc} B(G, G, Y) & \xrightarrow{\epsilon(\alpha)} & Y \\ \downarrow \pi & & \downarrow p \\ B(*, G, Y) & \xrightarrow{\epsilon(\bar{p})} & X \end{array}$$

Let  $b: * \rightarrow B(*, G, Y)$  be a point and let  $x = \epsilon(p) \circ b: * \rightarrow X$ . The map of fibers

$$\epsilon(\alpha): B(G, G, Y)_b \rightarrow Y_x.$$

is isomorphic to the composite

$$G \cong G \boxtimes * \xrightarrow{\text{id} \boxtimes y} G \boxtimes (G^{\boxtimes q} \boxtimes Y_x) \xrightarrow{\alpha} Y_x,$$

where  $y: * \rightarrow G^{\boxtimes q} \boxtimes Y_x$  is determined by a choice of representative

$$(t, y) \in \Delta^q \times B_q(*, G, Y)$$

for  $b$  at some simplicial level. Thus Lemma 4.5 implies that  $\epsilon(\alpha)$  induces a  $q$ -equivalence of fibers.

The projection  $\pi$  is a quasifibration by Proposition 3.10. Since  $p$  is an  $h$ -fibration of  $G$ -modules, by neglect of structure it is a level  $h$ -fibration, hence a level quasifibration. It follows that  $p$  is also a quasifibration of  $\mathcal{I}$ -spaces. This implies that  $\epsilon(\alpha)$  induces a  $q$ -equivalence of homotopy fibers. The map  $\epsilon(\alpha)$  is a homotopy equivalence of  $\mathcal{I}$ -spaces, so the five lemma implies that  $\epsilon(\bar{p})$  is a  $q$ -equivalence.  $\square$

We are now ready to prove Theorem 4.1. We will construct natural inverse maps  $\Psi: [X, B'G] \rightarrow \mathcal{E}_G(X)$  and  $\Phi: \mathcal{E}_G(X) \rightarrow [X, B'G]$ . Let us denote  $q$ -fibrant approximation by  $(-)'$ , where we take  $q$ -fibrant approximation in the category of  $G$ -modules when that structure is present. Write  $B'(X, G, Y)$  for the  $q$ -fibrant approximation of the bar construction  $B(X, G, Y)$ , and make the abbreviations  $B'G = B'(*, G, *)$  and  $E'G = B'(G, G, *)$ . Although we have no control over the fiber of  $\pi': E'G \rightarrow B'G$ , we know that its homotopy fiber is  $q$ -equivalent to  $G$  because  $\pi$  is a quasifibration. It follows that  $\Gamma\pi': \Gamma E'G \rightarrow B'G$  is a principal  $G$ -fibration. Given  $[f] \in [X, B'G]$ , define  $\Psi[f] = (f^* \Gamma E'G, \Gamma\pi')$ , the pullback of the principal  $G$ -fibration  $\Gamma E'G \rightarrow B'G$  along  $f$ . The map  $\Psi$  is well-defined by Proposition 4.3.

Now suppose that  $p: Y \rightarrow X$  is a principal  $G$ -fibration. Define the classifying map  $\Phi(Y, p): X \rightarrow B'G$  as follows. In the following commutative diagram,

$$\begin{array}{ccccc} X & \xleftarrow{\epsilon(\overline{p})} & B(*, G, Y) & \xrightarrow{q} & BG \\ \downarrow r & & \downarrow r & & \downarrow r \\ X' & \xleftarrow[\substack{\text{---} \text{ } s \text{ ---}}]{\epsilon(\overline{p})'} & B'(*, G, Y) & \xrightarrow{q'} & B'G \end{array}$$

the map  $q$  is the projection off of  $Y$  and the vertical maps are all fibrant approximations. The resulting map  $\epsilon(\overline{p})'$  is a  $q$ -equivalence by Proposition 4.6, so by Lemma 4.2.(ii) we may choose a map  $s: X' \rightarrow B'(*, G, Y)$  such that  $\epsilon(\overline{p})' \circ s$  is homotopic to  $\text{id}_{X'}$ . Define  $\Phi(Y, p) = [q' \circ s \circ r]$ . The function  $\Phi$  is well-defined because  $s$  is unique up to homotopy.

We will now prove that  $\Psi\Phi = \text{id}$ . Given  $(Y, p) \in \mathcal{E}_G(X)$ , we have the classifying map  $f = q' \circ s \circ r$ , and we may choose a homotopy  $H: X \times I \rightarrow X$  from  $\epsilon(\overline{p})' \circ s$  to  $\text{id}_{X'}$ . Write  $g = q' \circ s$  and form the following diagram.

(4.1)

$$\begin{array}{ccccc} \Gamma B'(G, G, Y) & \xrightarrow{\Gamma q'} & \Gamma E'G & & \\ \swarrow \Gamma \epsilon(\alpha)' & \downarrow \tilde{H}_1 & \swarrow \tilde{s} & & \swarrow \tilde{g} \\ \Gamma Y' & \xleftarrow{\quad} & s^* \Gamma B'(G, G, Y) & \xrightarrow{K} & g^* \Gamma E'G \\ \downarrow \Gamma p' & \downarrow \Gamma \pi' & \downarrow s^* \Gamma \pi' & \downarrow \Gamma \pi' & \downarrow g^* \Gamma \pi' \\ & B'(*, G, Y) & \xrightarrow{q'} & B'G & \\ \swarrow \epsilon(\overline{p})' & \swarrow s & & \swarrow g & \\ X' & \xleftarrow{\quad} & X' & \xleftarrow{\quad} & X' \end{array}$$

The squares on the right that face into the page are both pullback squares and  $K$  is the induced map of pullbacks. The lower triangle commutes up to homotopy via  $H$  and the CHP for the  $h$ -fibration of  $G$ -modules  $\Gamma p': \Gamma Y' \rightarrow X'$  allows us to lift  $H$  to a homotopy  $\tilde{H}$  of  $G$ -modules that makes the upper triangle commute up to homotopy. The rest of the diagram commutes strictly.

The map  $\Gamma \epsilon(\alpha)'$  is a  $q$ -equivalence because  $\epsilon(\alpha)$  is a  $q$ -equivalence, and  $\tilde{s}$  is a  $q$ -equivalence because it is a map of  $h$ -fibrations that induces a  $q$ -equivalence on fibers and on base spaces. It follows that the map  $\tilde{H}_1$  is a  $q$ -equivalence as well. To analyze the right side of the diagram, first notice that in the diagram

$$\begin{array}{ccc} B(G, G, Y) & \xrightarrow{q} & EG \\ \pi \downarrow & & \downarrow \pi \\ B(*, G, Y) & \longrightarrow & BG \end{array}$$

the upper map  $q$  induces an isomorphism on fibers. Since the projections  $\pi$  are quasifibrations, the map  $q$  is a  $q$ -equivalence on homotopy fibers as well. Taking fibrant approximation of the entire diagram, the induced map of total objects  $q': B'(G, G, Y) \rightarrow E'G$  is a  $q$ -equivalence on homotopy fibers as well. Applying the approximation functor  $\Gamma$ , we see that  $\Gamma q'$  is also a  $q$ -equivalence on homotopy

fibers. Since its domain and codomain are  $h$ -fibrations, this means that  $\Gamma q'$  induces a  $q$ -equivalence on fibers. Returning to diagram (4.1), we see that the induced map of pullbacks  $K$  is also a  $q$ -equivalence on fibers. Since  $K$  is a map of  $h$ -fibrations over the same base object, this means that  $K$  is a  $q$ -equivalence.

We have now established that the front wall of diagram (4.1) displays a chain of  $q$ -equivalences of principal  $G$ -fibrations between  $(\Gamma Y', \Gamma p')$  and  $(g^* \Gamma E' G, g^* \Gamma \pi')$ . Taking the pullback of these  $q$ -equivalences along the  $q$ -fibrant approximation map  $r: X \rightarrow X'$  gives a chain of  $q$ -equivalences between  $(r^* \Gamma Y', r^* \Gamma p')$  and  $\Psi \Phi(Y, p) = (f^* \Gamma E' G, f^* \Gamma \pi')$ . A little diagram chase shows that the induced map  $Y \rightarrow r^* \Gamma Y'$  is a  $q$ -equivalence, so that the original principal  $G$ -fibration  $(Y, p)$  is equivalent to the pullback  $(r^* \Gamma Y', r^* \Gamma p')$ . This proves that  $\Psi \Phi = \text{id}$ .

To show that  $\Phi \Psi = \text{id}$ , let  $f: X \rightarrow B'G$  and consider the following commutative diagram:

$$\begin{array}{ccccc}
 X & \xrightarrow{r} & X' & \xleftarrow[\substack{\sim \\ s_1}]{\epsilon(\overline{f^* \Gamma \pi'})'} & B'(*, G, f^* \Gamma E' G) \\
 \downarrow f & & \downarrow f' & & \downarrow B'(\text{id}, \text{id}, \tilde{f}) \\
 B'G & \xrightarrow{r} & B''G & \xleftarrow[\substack{\sim \\ s_2}]{\epsilon(\overline{\Gamma \pi'})'} & B'(*, G, \Gamma E' G) \xrightarrow{q'} B'G
 \end{array}$$

The left square is a naturality square for fibrant approximation and the middle square is the result of taking the fibrant approximation of the corresponding square that relates the maps  $\epsilon(\overline{f^* \Gamma \pi'})'$  and  $\epsilon(\overline{\Gamma \pi'})'$  via  $f$  and  $\tilde{f}$ . The existence of the section up to homotopy  $s_1$  is part of the construction  $\Phi$ . Applying Proposition 4.6 to the principal  $G$ -fibration  $\Gamma \pi': \Gamma E' G \rightarrow B'G$ , we see that the map  $\epsilon(\overline{\Gamma \pi'})'$  is a  $q$ -equivalence. Thus we may also find a section  $s_2$  up to homotopy as indicated. The diagram involving  $s_1$  and  $s_2$  commutes up to homotopy by the uniqueness statement in Lemma 4.2.(ii). Since  $\Gamma E' G$  is contractible, the lower instance of  $q'$  is a  $q$ -equivalence. By Lemma 4.2.(i), the map

$$\Phi \Psi: [f] \mapsto [q' \circ s_1 \circ r] = [(q' \circ s_2 \circ r) \circ f]$$

is an automorphism of  $[X, B'G]$ . Thus  $\Psi$  is injective, so the identity  $\Psi = (\Psi \Phi) \Psi = \Psi(\Phi \Psi)$  implies that  $\Phi \Psi = \text{id}$ . This concludes the proof of Theorem 4.1.

## 5. MODEL CATEGORIES OF PARAMETRIZED DIAGRAM SPACES

In this section we discuss model category structures on parametrized  $\mathcal{I}$ -spaces. In the following section, we will relate these model structures with model category structures on parametrized spectra. We first recall some basic material from May-Sigurdsson [18].

The category  $\mathcal{K}$  of  $k$ -spaces admits a compactly generated topological model structure whose weak equivalences are the weak homotopy equivalences and whose fibrations are the Serre fibrations. We refer to this model structure as the  $q$ -model structure, and call its weak equivalences and fibrations the  $q$ -equivalences and  $q$ -fibrations. Let  $B$  be a compactly generated topological space. The category  $\mathcal{K}/B$  of spaces  $(X, p) = (p: X \rightarrow B)$  over  $B$  admits a model structure whose weak equivalences and fibrations are detected by the forgetful functor  $(X, p) \mapsto X$  to the  $q$ -model structure on  $\mathcal{K}$ . An ex-space is a space  $(X, p)$  over  $B$  along with a map  $s: B \rightarrow X$  such that  $p \circ s = \text{id}_B$ . The category  $\mathcal{K}_B$  of ex-spaces  $(X, p, s)$  also admits a model structure given by the forgetful functor to the  $q$ -model structure on

$\mathcal{K}$ . We refer to these model structures as the  $q$ -model structure on  $\mathcal{K}/B$  and  $\mathcal{K}_B$ , respectively. While both of these model structures are compactly generated and topological, they are not well-grounded, in the sense of [18, §5.3-5.6]. The problem is that the generating  $q$ -cofibrations and acyclic  $q$ -cofibrations do not satisfy the CHP defined in terms of the cylinder objects native to their category—they are only Hurewicz cofibrations in the underlying category of spaces. As a result, applications of the glueing lemma that would allow standard inductive arguments over cell complexes built out of the generating sets fail for these model structures. In attempting to construct a stable model structure on parametrized spectra based on the  $q$ -model structure, the verification that relative cell complexes built out of the generating acyclic cofibrations are weak equivalences is unattainable.

As an alternative, May-Sigurdsson develop the  $qf$ -model structure on  $\mathcal{K}/B$  and  $\mathcal{K}_B$ . The  $qf$ -model structure also has the  $q$ -equivalences as weak equivalences, so that the associated homotopy category is still the homotopy category of spaces over  $B$ , but there are fewer  $qf$ -cofibrations than  $q$ -cofibrations. A  $qf$ -fibration need not be a Serre fibration but will be a quasifibration of total spaces. For our purposes, we do not need the details of the definitions, only the fact that in each case the  $qf$ -model structure is a well-grounded compactly generated model category. We will work in the un-sectioned context, building well-grounded compactly generated model structures on parametrized diagram spaces out of the  $qf$ -model structure on  $\mathcal{K}/B$ .

We will now introduce the analogous  $qf$  model structure on parametrized  $\mathcal{I}$ -spaces. There is also a  $q$  model structure on parametrized  $\mathcal{I}$ -spaces, but we will not need to use it. When  $B$  is a point, both the  $q$  and  $qf$  model structure on parametrized  $\mathcal{I}$ -spaces agrees with the model structure on  $\mathcal{I}$ -spaces constructed in [12, §15].

We always assume that the base space  $B$  is a compactly generated topological space. By considering  $B$  as a constant  $\mathcal{I}$ -space  $B(V) = B$ , the category  $\mathcal{I}\mathcal{K}/B$  of  $\mathcal{I}$ -spaces over  $B$  may be identified with the category of continuous functors from  $\mathcal{I}$  to the category  $\mathcal{K}/B$  of spaces over  $B$ . As such, there is a level  $qf$  model structure on  $\mathcal{I}$ -spaces over  $B$  with weak equivalences and fibrations those maps  $X \rightarrow Y$  for which  $X(V) \rightarrow Y(V)$  is a weak homotopy equivalence or  $qf$ -fibration, respectively, in the  $qf$ -model structure on  $\mathcal{K}/B$  for every object  $V$  of  $\mathcal{I}$ . Define a  $qf$ -fibration of  $\mathcal{I}$ -spaces over  $B$  to be a map that has the right lifting property with respect to level-wise  $qf$ -cofibrations that are  $q$ -equivalences of  $\mathcal{I}$ -spaces. Define a  $qf$ -cofibration to be a level-wise  $qf$ -cofibration. Replacing  $\mathcal{U}$  under the  $q$ -model structure with  $\mathcal{K}/B$  under the  $qf$ -model structure, we may make the same arguments as in [12, §15] to prove the following theorem.

**Theorem 5.1.** *Let  $B$  be a compactly generated space.*

- (i) *The category  $\mathcal{I}\mathcal{K}/B$  of  $\mathcal{I}$ -spaces over  $B$  is a well-grounded compactly generated model category with respect to the  $q$ -equivalences,  $qf$ -fibrations and  $qf$ -cofibrations. We refer to this model structure as the  $qf$ -model structure on  $\mathcal{I}\mathcal{K}/B$ .*
- (ii) *Let  $G$  be an  $\mathcal{I}$ -FCP. There is a well-grounded compactly generated model structure on the category  $\text{Mod}_G/B$  of  $G$ -modules over  $B$  with weak equivalences and fibrations created by the forgetful functor to the  $qf$ -model structure on  $\mathcal{I}$ -spaces over  $B$ .*

It is a formal consequence that the category  $\mathcal{IK}_B$  of ex- $\mathcal{I}$ -spaces inherits a well-grounded compactly generated model structure from the  $qf$ -model structure on  $\mathcal{IK}/B$ . We will pass through this category briefly when constructing parametrized spectra, but we will not need to do any real work there.

It will be useful to have the following description of the  $qf$ -fibrations.

**Proposition 5.2.** *A map  $(X, p) \rightarrow (Y, q)$  of  $\mathcal{I}$ -spaces over  $B$  is a  $qf$ -fibration if and only if it is a level-wise  $qf$ -fibration and for every morphism  $\varphi: V \rightarrow W$  of  $\mathcal{I}$ , the induced map to the pullback*

$$X_V \rightarrow X_W \times_{Y_W} Y_V$$

*is a  $q$ -equivalence of spaces over  $B$ . In particular,  $(X, p)$  is  $qf$ -fibrant if and only if each structure map  $p_V: X_V \rightarrow B$  is a  $qf$ -fibration of spaces and every morphism  $\varphi: V \rightarrow W$  of  $\mathcal{I}$  induces a  $q$ -equivalence  $X_V \rightarrow Y_W$  of spaces over  $B$ . Since  $qf$ -fibrations of spaces are quasifibrations and level-wise quasifibrations of  $\mathcal{I}$ -spaces are quasifibrations of  $\mathcal{I}$ -spaces, it follows that every  $qf$ -fibration of  $\mathcal{I}$ -spaces is a quasifibration of  $\mathcal{I}$ -spaces.*

While the following terminology is non-standard, it will be useful as a halfway house relating the highly structured notion of a principal  $G$ -fibration with the model-theoretic fiber conditions on parametrized spectra.

**Definition 5.3.** A  $G$ -torsor over  $B$  is a  $G$ -module  $(Y, p)$  over  $B$  whose derived fibers are weakly equivalent to  $G$ . In other words, given a  $qf$ -fibrant approximation  $Y'$  of  $Y$  in the category of  $G$ -modules over  $B$ , each fiber admits a chain of  $q$ -equivalences of  $G$ -modules  $Y'_b \simeq G$

The inclusion of the fiber into the homotopy fiber for a  $G$ -module  $Y$  over  $B$  and a  $qf$ -fibrant approximation  $Y'$  are related by the commutative diagram

$$(5.1) \quad \begin{array}{ccc} Y_b & \longrightarrow & Y'_b \\ \downarrow & & \downarrow \simeq \\ F_b(p) & \xrightarrow{\simeq} & F_b(p') \end{array}$$

induced by fibrant approximation. Since the fibrant approximation is a  $q$ -equivalence of total spaces, it induces a  $q$ -equivalence of the homotopy fibers. The  $qf$ -fibration  $p'$  is in particular a quasifibration of  $\mathcal{I}$ -spaces, which gives the other displayed  $q$ -equivalence. We then have the following characterization of  $G$ -torsors.

**Lemma 5.5.** *A  $G$ -module  $(Y, p)$  over  $B$  is a  $G$ -torsor if and only if for every  $b \in B$ , the homotopy fiber  $F_b(p)$  admits a chain of  $q$ -equivalences of  $G$ -modules  $F_b(p) \simeq G$ .*

From now on we let  $F_b(-)$  denote the derived fiber functor on  $G$ -modules over  $B$ , meaning that  $F_b(Y)$  is the fiber over  $b$  of a  $qf$ -fibrant approximation of  $Y$ . This potentially confusing notation will agree with notation for the derived fiber of parametrized spectra and is justified by the chain of  $q$ -equivalences in diagram (5.1).

Let  $\text{Ho}(G\text{Mod}/B)$  denote the homotopy category of  $G$ -modules over  $B$  formed using the  $qf$ -model structure on  $G$ -modules over  $B$ . Let  $\text{Ho}(G\text{Tor}/B)$  be the subcategory of  $\text{Ho}(G\text{Mod}/B)$  consisting of  $G$ -torsors and  $q$ -equivalences of  $G$ -torsors. Recall the  $h$ -fibration approximation functor  $\Gamma$  from §4. Using the characterization of  $G$ -torsors in Lemma 5.5, it follows from Proposition 4.4 that  $\Gamma$  takes  $G$ -torsors

to principal  $G$ -fibrations and preserves  $q$ -equivalences. Conversely, every principal  $G$ -fibration satisfies the condition in Lemma 5.5 and thus is a  $G$ -torsor. The map  $\eta: Y \rightarrow \Gamma Y$  in Proposition 4.4.(iii) is a  $q$ -equivalence of  $G$ -modules, so the next result follows.

**Proposition 5.6.** *The functor  $\Gamma$  induces a natural isomorphism between the set  $\pi_0 \text{Ho}(G \text{ Tor}/B)$  of isomorphism classes of  $G$ -torsors over  $B$  in the homotopy category and the set  $\mathcal{E}_G(B)$  of equivalence classes of principal  $G$ -fibrations over  $B$ .*

## 6. MODEL CATEGORIES OF PARAMETRIZED SPECTRA

We will now summarize what we need from the theory of parametrized spectra, following chapters 11 and 12 of May-Sigurdsson [18]. A spectrum over  $B$  is an orthogonal spectrum in the category of ex-spaces over  $B$ . In other words, a parametrized spectrum  $X$  consists of an  $O(V)$ -equivariant ex-space  $(X(V), p(V), s(V))$  for each finite dimensional real inner product space  $V$ , along with compatible equivariant structure maps

$$\sigma: X(V) \wedge_B S_B^W \rightarrow X(V \oplus W)$$

over and under  $B$ . Here  $S_B^V = r^* S^V = S^V \times B$  is the trivially twisted ex-space with fiber the one-point compactification  $S^V$ . The section of  $S_B^V$  is determined by the basepoint of  $S^V$ . The smash product  $\wedge_B$  is the fiberwise smash product of ex-spaces. A map  $f: X \rightarrow Y$  of spectra over  $B$  consists of an equivariant map  $f(V): X(V) \rightarrow Y(V)$  of ex-spaces for each indexing space  $V$  that are suitably compatible with the structure maps  $\sigma$ . For each point  $b \in B$ , the fiber of  $X$  over  $b$  is the spectrum  $X_b = i_b^* X$  given by the pullback of  $X$  along the base change functor associated to the inclusion map  $i_b: \{b\} \rightarrow B$ . The fiber spectrum is described level-wise in terms of the fibers of its constituent ex-spaces by the formula  $X_b(V) = X(V)_b$ .

The level model structure on the category  $\mathcal{S}_B$  of spectra over  $B$  has as weak equivalences, respectively fibrations, those maps  $f$  such that each  $f(V)$  is a  $q$ -equivalence, respectively  $qf$ -fibration, of ex-spaces. We refer to these maps as the level-wise  $q$ -equivalences and level-wise  $qf$ -fibrations, respectively. The homotopy groups of a level-wise  $qf$ -fibrant spectrum  $X$  over  $B$  are the homotopy groups  $\pi_q X_b$  of all of the fibers of  $X$ . The homotopy groups of a spectrum  $X$  over  $B$  are the homotopy groups  $\pi_q(R_l X)_b$  of the fibers of a level-wise  $qf$ -fibrant approximation  $R_l X$  of  $X$ . We say that a map  $X \rightarrow Y$  of spectra over  $B$  is a stable equivalence if it induces an isomorphism on all homotopy groups of all fibers. An  $\Omega$ -spectrum over  $B$  is a level  $qf$ -fibrant spectrum  $X$  over  $B$  whose adjoint structure maps

$$\tilde{\sigma}: X(V) \rightarrow \Omega_B^W X(V \oplus W)$$

are  $q$ -equivalences of ex-spaces over  $B$ . By looking at the induced map on fibers, a level  $q$ -equivalence of spectra is a stable equivalence. We can now describe the stable model structure on parametrized spectra.

**Theorem 6.1.** [18, 12.3.10] *The category  $\mathcal{S}_B$  of spectra over  $B$  is a well-grounded compactly generated model category with respect to the stable equivalences, the  $s$ -fibrations and the  $s$ -cofibrations. Furthermore, the  $s$ -fibrant objects are the  $\Omega$ -spectra over  $B$ . We refer to this model structure as the  $s$ -model structure (or stable model structure) on  $\mathcal{S}_B$ .*

In the case  $B = *$ , we recover the stable model structure on orthogonal spectra from Mandell-May-Schwede-Shipley [14].

Parametrized  $\mathcal{I}$ -spaces and parametrized spectra are related by suspension spectrum and underlying infinite loop space functors. If  $(Y, p)$  is an  $\mathcal{I}$ -space over  $B$ , the suspension spectrum  $\Sigma_B^\bullet Y$  is the spectrum over  $B$  defined by

$$(\Sigma_B^\bullet Y)(V) = (Y(V), p)_B \wedge_B S_B^V,$$

where  $(Y(V), p)_B$  is the ex-space over  $B$  obtained from  $(Y(V), p)$  by adjoining a disjoint section:

$$(Y(V), p)_B = (Y(V) \amalg B, p \amalg \text{id}_B, \text{id}_B).$$

The spectrum structure maps  $\sigma$  are defined using the map  $Y(V) \rightarrow Y(V \oplus W)$  of ex-spaces induced by the canonical inclusion  $V \rightarrow V \oplus W$  and the canonical isomorphism  $S_B^V \wedge_B S_B^W \cong S_B^{V \oplus W}$ . When  $B = *$ , this agrees with the definition in [12] of the suspension spectrum functor  $\Sigma_+^\bullet$  carrying  $\mathcal{I}$ -spaces to spectra. Notice that we define  $\Sigma_B^\bullet$  to take un-sectioned  $\mathcal{I}$ -spaces as input. In other words,  $\Sigma_B^\bullet$  is the parametrized analog of  $\Sigma_+^\bullet$ . This is in notational conflict with May-Sigurdsson's use of  $\Sigma_B^\infty$  to denote the suspension spectrum of an ex-space, no disjoint section added.

If  $X$  is a spectrum over  $B$ , we define the  $\mathcal{I}$ -space  $\Omega_B^\bullet X$  over  $B$  by

$$(\Omega_B^\bullet X)(V) = \Omega_B^V X(V) = F_B(S_B^V, X(V)).$$

Here  $F_B(Y, Z)$  is the ex-space of fiberwise based maps  $Y \rightarrow Z$ . The fiber  $F_B(Y, Z)_b$  over each point  $b \in B$  consists of the space of based maps  $Y_b \rightarrow Z_b$ , and  $F_B(Y, -)$  is right adjoint to the fiberwise smash product  $(-) \wedge_B Y$ . The functoriality of  $\Omega_B^\bullet X$  in  $\mathcal{I}$  follows as in the non-parametrized context. These functors form an adjunction:

$$\mathcal{IK}/B \begin{matrix} \xrightarrow{\Sigma_B^\bullet} \\ \xleftarrow{\Omega_B^\bullet} \end{matrix} \mathcal{S}_B.$$

Inspection of the definitions gives natural isomorphisms of fibers

$$(\Sigma_B^\bullet Y)_b \cong \Sigma_+^\bullet Y_b$$

and

$$(\Omega_B^\bullet X)_b \cong \Omega^\bullet X_b.$$

The category  $\mathcal{IK}/B$  is enriched and tensored over the category  $\mathcal{IK}$  of  $\mathcal{I}$ -spaces with tensor given by the symmetric monoidal product  $\boxtimes$ . Similarly, the category  $\mathcal{S}_B$  of spectra over  $B$  is enriched and tensored over the category  $\mathcal{S}$  of spectra with tensor the fiberwise smash product  $\wedge$  (May-Sigurdsson denote this by  $\overline{\wedge}$ ). The adjunction  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  respects the enrichments and tensoring, in the following sense.

**Proposition 6.2.**

- (i) *Let  $A$  be an  $\mathcal{I}$ -space and let  $Y$  be an  $\mathcal{I}$ -space over  $B$ . There is a natural isomorphism of parametrized spectra over  $B$*

$$\Sigma_B^\bullet(A \boxtimes Y) \cong \Sigma_+^\bullet A \wedge \Sigma_B^\bullet Y$$

*that satisfies the analogs of the associativity and unit diagrams for a monoidal natural transformation.*

- (ii) Let  $D$  be a spectrum and let  $X$  be a parametrized spectrum over  $B$ . There is a natural transformation of  $\mathcal{I}$ -spaces over  $B$

$$\Omega^\bullet D \boxtimes \Omega_B^\bullet X \longrightarrow \Omega_B^\bullet(D \wedge X)$$

satisfying the analogs of the associativity and unit diagrams for a monoidal natural transformation.

We now turn to parametrized module spectra. Let  $R$  be a (non-parametrized) ring spectrum. We assume, once and for all, that  $R$  is well-grounded, meaning that each  $R(V)$  is well-based and compactly generated. An  $R$ -module over  $B$  is a spectrum  $N$  over  $B$  with an associative and unital map of spectra over  $B$

$$R \wedge N \longrightarrow N,$$

where  $\wedge$  denotes the tensor of a spectrum with a spectrum over  $B$ .

**Theorem 6.3.** [18, 14.1.7] *The category  $R\text{Mod}_B$  of  $R$ -modules over  $B$  is a well-grounded compactly generated model category with weak equivalences and fibrations created by the forgetful functor to  $\mathcal{S}_B$ . We refer to this model structure as the  $s$ -model structure on  $R\text{Mod}_B$ .*

It is a consequence of Proposition 6.2 that the adjunction  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  restricts to an adjunction between  $G$ -modules over  $B$  and  $\Sigma_+^\bullet G$ -module spectra over  $B$ . We now consider the homotopical properties of the adjunction  $(\Sigma_B^\bullet, \Omega_B^\bullet)$ .

**Proposition 6.4.**

- (i) *The adjoint pair  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  is a Quillen adjunction between the  $qf$ -model structure on  $\mathcal{I}$ -spaces over  $B$  and the  $s$ -model structure on spectra over  $B$ .*
- (ii) *Let  $G$  be an  $\mathcal{I}$ -FCP. The adjoint pair  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  is a Quillen adjunction between the  $qf$ -model structure on  $G$ -modules over  $B$  and the  $s$ -model structure on  $\Sigma_+^\bullet G$ -modules over  $B$ .*

*Proof.* By the descriptions of fibrant objects and fibrations in Theorem 6.1 and Proposition 5.2, (i) follows using the same proof as for the non-parametrized Quillen adjunction  $(\Sigma_+^\bullet, \Omega^\bullet)$  [12, 2.5 and 3.6]. The only difference is that  $qf$ -fibrations are level  $qf$ -fibrations instead of level  $q$ -fibrations. However,  $qf$ -fibrations of spaces over  $B$  are closed under pullbacks and are quasifibrations [18, 6.5.1], so we can still use the five lemma argument given in the cited proof. For the preservation of acyclic fibrations, use [18, 12.6.2] to deduce that an acyclic  $s$ -fibration of parametrized spectra is a level  $q$ -equivalence. Part (ii) follows because fibrations and acyclic fibrations of modules are detected in the underlying model structures in (i).  $\square$

It is a formal consequence that the left Quillen functor  $\Sigma_B^\bullet$  preserves weak equivalences between cofibrant objects. However, it will be useful to know that a stronger result is true.

**Lemma 6.5.** *The functors  $\Sigma_+^\bullet: \mathcal{IK} \longrightarrow \mathcal{S}$  and  $\Sigma_B^\bullet: \mathcal{IK}/B \longrightarrow \mathcal{S}_B$  preserve all weak equivalences.*

*Proof.* Let us write  $\text{hocolim}_m^*(-)$  for the based homotopy colimit over the poset category  $\mathcal{N}$  of natural numbers. Let  $X$  be an  $\mathcal{I}$ -space and write  $X(m) = X(\mathbf{R}^m)$ . By filtering the homotopy colimit by finite stages, we see that the natural map

$$\text{hocolim}_m^* \Omega^n \Sigma_+^n X(m) \longrightarrow \Omega^n \text{hocolim}_m^* \Sigma_+^n X(m)$$

is a weak homotopy equivalence. We now take the homotopy colimits over  $n$  and observe that the functor  $\Sigma_n^+$  converts unbased homotopy colimits to based homotopy colimits, giving a weak homotopy equivalence:

$$\operatorname{hocolim}_n^* \operatorname{hocolim}_m^* \Omega^n \Sigma_+^n X(m) \xrightarrow{\simeq} \operatorname{hocolim}_n^* \Omega^n \Sigma_+^n \operatorname{hocolim}_m X(m).$$

The inclusion of categories  $\mathcal{N} \rightarrow \mathcal{J} \rightarrow \mathcal{I}$  is homotopy cofinal, so by applying the homotopy cofinality criterion for topological homotopy colimits [12, A.5], the codomain is homotopy equivalent to  $\operatorname{hocolim}_n^* \Omega^n \Sigma_+^n X_{h\mathcal{I}}$ . On the other hand, the diagonal functor  $\mathcal{N} \rightarrow \mathcal{N} \times \mathcal{N}$  is also homotopy cofinal, so the domain is homotopy equivalent to  $\operatorname{hocolim}_n^* \Omega^n \Sigma_+^n X(n)$ . Since all of the equivalences are induced by natural transformations, it follows that if  $f$  is a  $q$ -equivalence of  $\mathcal{I}$ -spaces, then  $\operatorname{hocolim}_n^* \Omega^n \Sigma_+^n f(n)$  is a weak homotopy equivalence, which implies that  $\Sigma_+^\bullet f$  is a stable equivalence of orthogonal spectra.

We now deduce the result for  $\Sigma_B^\bullet$ . Assume that  $f: X \rightarrow Y$  is a  $q$ -equivalence of  $\mathcal{I}$ -spaces over  $B$ . We will use the notion of an ex-quasifibrant space over  $B$ ; this is an ex-space  $(Y, p, s)$  whose structure map  $p$  is a quasifibration and whose section  $s$  is an  $\bar{f}$ -cofibration, in the language of [18, 8.1.1, 8.5.1]. The functor  $\Sigma_B^\bullet$  preserves level-wise ex-quasifibrant objects by [18, 8.5.3.(iii)]. Taking a level-wise ex-quasifibrant approximation  $f': X' \rightarrow Y'$  of  $f$ , we see by [18, Lemma 12.4.1] that  $\Sigma_B^\bullet f$  is a stable equivalence if and only if the induced map of fibers of the approximation  $\Sigma_+^\bullet f'_b: \Sigma_+^\bullet (X')_b \rightarrow \Sigma_+^\bullet (Y')_b$  is a stable equivalence. Since the structure maps  $X' \rightarrow B$  and  $Y' \rightarrow B$  are level-wise quasifibrations, they are quasifibrations of  $\mathcal{I}$ -spaces. By the long exact sequence of Lemma 3.6, the  $q$ -equivalence  $f': X' \rightarrow Y'$  induces a  $q$ -equivalence of fibers  $f'_b: (X')_b \rightarrow (Y')_b$ . The lemma now follows since  $\Sigma_+^\bullet$  preserves weak equivalences.  $\square$

We will work in the non-parametrized setting for a moment in order to fix notation on some constructions. Suppose that  $R$  and  $A$  are ring spectra. Consider the function spectrum  $F^R(-, -)$  of  $R$ -modules as defined in [14, diagram (22.3)]. If  $P$  is an  $A$ -module,  $M$  is an  $(R, A)$ -bimodule and  $N$  is an  $R$ -module, then  $F^R(M, N)$  is an  $A$ -module and we have the following adjunction:

$$(6.1) \quad \operatorname{Mod}_R(M \wedge_A P, N) \cong \operatorname{Mod}_A(P, F^R(M, N)).$$

The following invariance result is a consequence of the fact that the category of  $R$ -modules is a spectrally enriched model category via the function spectra  $F^R(-, -)$ .

**Lemma 6.7.** *If  $M'$  is a cofibrant  $R$ -module, then the functor  $F^R(M', -)$  preserves stable equivalences between fibrant  $R$ -modules. If  $N$  is a fibrant  $R$ -module, then the functor  $F^R(-, N)$  preserves stable equivalences between cofibrant  $R$ -modules.*

We will be interested in the generalization of the adjunction (6.1) where  $N$  and  $P$  are parametrized spectra. The smash product  $M \wedge_A P$  occurring in the parametrized version of the adjunction is built out of the external smash product  $\bar{\wedge}: \mathcal{S} \times \mathcal{S}_B \rightarrow \mathcal{S}_B$ , as described in [18, §14.1]. In particular, there is never a need to internalize the smash product by taking the pullback  $\Delta^*$  of a spectrum over  $B \times B$  along the diagonal map. As May-Sigurdsson explain, we are able to maintain homotopical control of the smash product  $\wedge_A$  in this situation.

**Lemma 6.8.** *Suppose that  $R$  and  $A$  are  $s$ -cofibrant ring spectra. Let  $i: X \rightarrow Y$  be an  $s$ -cofibration of  $(R, A)$ -bimodules and let  $j: Z \rightarrow W$  be an  $s$ -cofibration of  $A$ -modules over  $B$ . Then the pushout product*

$$i \square j: (Y \wedge_A W) \cup_{X \wedge_A W} (X \wedge_A Z) \rightarrow Y \wedge_A Z$$

*is an  $s$ -cofibration of  $R$ -modules over  $B$  which is a stable equivalence if either  $i$  or  $j$  is.*

*Proof.* Using the fact that all of the module categories are well-grounded, we may induct up the cellular filtration of  $i$  and  $j$ , so it suffices to verify the result when  $i$  and  $j$  are generating cofibrations or generating acyclic cofibrations.

For generating maps, the extra factor of  $A$  on either side cancels with the smash product over  $A$ , so we may deduce the result from the corresponding result for  $A = S$  [18, 12.6.5], as in the proof of [21, 4.1.(2)]. We use the fact that  $A$  and  $R$  are  $s$ -cofibrant spectra to insure that  $A \wedge (-)$  preserves  $s$ -cofibrations and acyclic  $s$ -cofibrations of spectra over  $B$  and that  $R \wedge (-)$  takes  $s$ -cofibrations and acyclic  $s$ -cofibrations to  $s$ -cofibrations and acyclic  $s$ -cofibrations of  $R$ -modules.  $\square$

The lemma has the following consequence.

**Proposition 6.9.** *Let  $A$  and  $R$  be  $s$ -cofibrant ring spectra. Suppose that  $M$  is a cofibrant  $(R, A)$ -bimodule. Then the adjunction*

$$(A\text{-modules over } B) \begin{array}{c} \xleftarrow{M \wedge_A (-)} \\ \xrightarrow{F^R(M, -)} \end{array} (R\text{-modules over } B)$$

*is a Quillen adjunction.*

## 7. THE PRINCIPAL $\text{Aut}_R M$ -FIBRATION ASSOCIATED TO AN $R$ -BUNDLE

Let  $R$  be a ring spectrum and let  $M$  be an  $R$ -module. We assume from now on that  $R$  is an  $s$ -cofibrant ring spectrum and that  $M$  is both  $s$ -fibrant and  $s$ -cofibrant as an  $R$ -module. In this section, we will define the  $\mathcal{I}$ -FCP  $\text{Aut}_R M$  of equivalences of  $R$ -modules  $M \rightarrow M$ . We then describe the construction of an  $\text{Aut}_R M$ -torsor from an  $R$ -bundle with fiber  $M$ , as well as the inverse construction of an  $R$ -bundle with fiber  $M$  from an  $\text{Aut}_R M$ -torsor over  $B$ . In the following section, we will pass to homotopy categories and show that the induced derived functors give a bijection between the sets of equivalence classes of each type of object. Since principal  $\text{Aut}_R M$ -fibrations are equivalent to  $\text{Aut}_R M$ -torsors, this will give the equivalence described in the introduction between principal  $\text{Aut}_R M$ -fibrations and  $R$ -bundles with fiber  $M$ . In order to maintain homotopical control, we will actually work with a cofibrant approximation  $\text{Aut}_R^c M$  of  $\text{Aut}_R M$ .

The function spectrum  $F^R(M, M)$  is a ring spectrum under composition of maps. Let  $\text{End}_R M = \Omega^\bullet F^R(M, M)$  be the underlying  $\mathcal{I}$ -FCP. We define  $\text{Aut}_R M$  to be the  $\mathcal{I}$ -FCP of units of the ring spectrum  $F^R(M, M)$ :

$$\text{Aut}_R M = \text{GL}_1^\bullet F^R(M, M).$$

Here we follow [12, §12] in defining the  $\mathcal{I}$ -FCP of units  $\mathrm{GL}_1^\bullet A$  of a ring spectrum  $A$  to be the following pullback of  $\mathcal{I}$ -FCPs

$$\begin{array}{ccc} \mathrm{GL}_1^\bullet A & \longrightarrow & \Omega^\bullet A \\ \downarrow & & \downarrow \\ (\pi_0 A)^\times & \longrightarrow & \pi_0 A \end{array}$$

In particular  $\mathrm{Aut}_R M$  is naturally a sub- $\mathcal{I}$ -FCP of  $\mathrm{End}_R M$  and is given level-wise by the inclusion of those path components that are stably invertible under the FCP multiplication. We think of  $\mathrm{Aut}_R M$  as the  $A_\infty$  space of equivalences of  $R$ -modules  $M \rightarrow M$ . The  $\mathcal{I}$ -FCP structure on  $\mathrm{Aut}_R M$  gives rise to a ring spectrum  $\Sigma_+^\bullet \mathrm{Aut}_R M$ . The  $R$ -module  $M$  also has the structure of a right  $\Sigma_+^\bullet \mathrm{Aut}_R M$ -module, with action map

$$M \wedge_S \Sigma_+^\bullet \mathrm{Aut}_R M \rightarrow M$$

the adjoint of the composite map of ring spectra

$$\Sigma_+^\bullet \mathrm{Aut}_R M \rightarrow \Sigma_+^\bullet \Omega^\bullet F_R(M, M) \xrightarrow{\epsilon} F_R(M, M)$$

induced by the canonical inclusion  $\mathrm{GL}_1^\bullet \rightarrow \Omega^\bullet$  and the counit of the adjunction  $(\Sigma_+^\bullet, \Omega^\bullet)$ . Thus  $M$  is an  $(R, \Sigma_+^\bullet \mathrm{Aut}_R M)$ -bimodule.

Let  $\mathrm{Aut}_R^c M \rightarrow \mathrm{Aut}_R M$  be a  $q$ -cofibrant approximation of  $\mathrm{Aut}_R M$  as an  $\mathcal{I}$ -FCP. The right  $\Sigma_+^\bullet \mathrm{Aut}_R M$ -module structure of  $M$  pulls back to give a right  $\Sigma_+^\bullet \mathrm{Aut}_R^c M$ -module structure on  $M$ . By identifying  $R \wedge_S (\Sigma_+^\bullet \mathrm{Aut}_R^c M)^{\mathrm{op}}$ -modules with  $(R, \Sigma_+^\bullet \mathrm{Aut}_R^c M)$ -bimodules, the category of  $(R, \Sigma_+^\bullet \mathrm{Aut}_R^c M)$ -bimodules is a well-grounded compactly generated model category with weak equivalences and fibrations created in the  $s$ -model structure on spectra [14, 12.1]. Let  $M^\circ \rightarrow M$  be an  $s$ -cofibrant approximation of  $M$  as an  $(R, \Sigma_+^\bullet \mathrm{Aut}_R^c M)$ -bimodule. Note that since  $\Sigma_+^\bullet$  is left Quillen,  $\Sigma_+^\bullet \mathrm{Aut}_R^c M$  is  $s$ -cofibrant as a ring spectrum, and thus  $s$ -cofibrant as a spectrum. We record a basic consequence.

**Lemma 7.1.** *The underlying left  $R$ -module of  $M^\circ$  is  $s$ -cofibrant. The underlying right  $\Sigma_+^\bullet \mathrm{Aut}_R^c M$ -module of  $M^\circ$  is  $s$ -cofibrant.*

*Proof.* The right adjoint of the forgetful functor from  $(R, \Sigma_+^\bullet \mathrm{Aut}_R^c M)$ -bimodules to left  $R$ -modules is the function spectrum functor  $F^S(\Sigma_+^\bullet \mathrm{Aut}_R^c M, -)$ . This functor preserves fibrations and acyclic fibrations because  $\Sigma_+^\bullet \mathrm{Aut}_R^c M$  is  $s$ -cofibrant. Therefore its left adjoint the forgetful functor preserves cofibrations and acyclic cofibrations. This proves the first claim. The second claim follows using a similar argument and the fact that  $R$  is  $s$ -cofibrant.  $\square$

Let  $N$  be an  $R$ -module over  $B$ . We will write  $F_b N$  for the fiber  $i_b^* R N$  of an  $s$ -fibrant approximation of  $N$  and write  $\mathbf{F}_b = \mathbf{R}i_b^*(-)$  for the right derived fiber functor.

**Definition 7.2.** An  $R$ -bundle over  $B$  with fiber  $M$  is an  $R$ -module  $N$  over  $B$  such that every derived fiber  $\mathbf{F}_b N$  of  $N$  admits a zig-zag of stable equivalences of  $R$ -modules to  $M$ . A map of  $R$ -bundles over  $B$  with fiber  $M$  is a map of  $R$ -modules over  $B$ . A stable equivalence of  $R$ -bundles is a map of  $R$ -bundles that is a stable equivalence of parametrized spectra.

Let  $N$  be an  $R$ -bundle over  $B$ . The function spectrum  $F_B^R(M^\circ, N)$  is a  $\Sigma_+^\bullet \text{End}_R M$ -module over  $B$ . Applying  $\Omega_B^\bullet$ , we get an  $\text{End}_R M$ -module  $\Omega_B^\bullet F_B^R(M^\circ, N)$  over  $B$ . The following lemma allows us to keep control of its fiber homotopy type.

**Lemma 7.3.** *Suppose that  $N$  is  $s$ -fibrant and fix a point  $b \in B$ . A stable equivalence of  $R$ -modules  $N_b \simeq M$  determines:*

(i) *a stable equivalence of  $\Sigma_+^\bullet \text{End}_R M$ -modules*

$$F^R(M^\circ, N)_b \simeq F^R(M, M);$$

(ii) *a  $q$ -equivalence of  $\text{End}_R M$ -modules*

$$\Omega_B^\bullet F^R(M^\circ, N)_b \simeq \Omega^\bullet F^R(M, M).$$

*Proof.* Fix a stable equivalence of  $R$ -modules  $N_b \simeq M$ . Since  $N$  is  $s$ -fibrant, the fiber  $N_b$  is also  $s$ -fibrant. The  $R$ -modules  $M$  and  $M^\circ$  are both  $s$ -cofibrant, so we have stable equivalences of function spectra

$$F^R(M^\circ, N_b) \simeq F^R(M, N_b) \simeq F^R(M, M).$$

There is a canonical isomorphism of the fiber of the function spectrum

$$F^R(M^\circ, N)_b \cong F^R(M^\circ, N_b),$$

so this gives (i). The  $q$ -equivalence in (ii) follows by applying  $\Omega^\bullet$  and using the canonical isomorphism of fibers  $\Omega_B^\bullet F^R(M^\circ, N)_b \cong \Omega^\bullet(F^R(M^\circ, N)_b)$ .  $\square$

Notice that the second condition in the lemma says that  $\Omega_B^\bullet F^R(M^\circ, N)$  is an  $\text{End}_R M$ -torsor. We will now construct a maximal  $\text{Aut}_R M$ -torsor

$$E^R(M^\circ, N) \subset \Omega_B^\bullet F^R(M^\circ, N).$$

The idea of the construction is to restrict to the sub-parametrized  $\mathcal{I}$ -space whose fiber over  $b \in B$  consists of the stable equivalences of  $R$ -modules  $M^\circ \rightarrow N_b$ . To make this idea rigorous, we need to access the components  $\pi_0 \Omega_B^\bullet F^R(M^\circ, N)_b$  of each fiber in a way that remembers the topology of  $B$ .

To this end, we will define the parametrized components  $\pi_0^B X$  of a parametrized space  $p: X \rightarrow B$ . As a set,  $\pi_0^B X$  consists of all components of all fibers of  $X$ :

$$\pi_0^B X = \bigcup_{b \in B} \pi_0 X_b.$$

Give  $\pi_0^B X$  the quotient topology induced by the map  $X \rightarrow \pi_0^B X$  that sends a point  $x \in X$  to its component  $[x] \in \pi_0 X_{p(x)}$ . Since the quotient map is a map over  $B$ , the space  $\pi_0^B X$  is a parametrized space over  $B$ .

Let  $G$  be an  $\mathcal{I}$ -FCP and let  $Y$  be a  $G$ -torsor over  $B$ . Assume that  $Y$  is  $qf$ -fibrant as a  $G$ -module over  $B$ . While  $G$  may not be grouplike, there is a maximal grouplike sub  $\mathcal{I}$ -FCP  $G^\times \subset G$ , as defined in [12, §12]. When  $G = \Omega^\bullet A$  is the underlying multiplicative  $\mathcal{I}$ -FCP of an orthogonal ring spectrum, the grouplike  $\mathcal{I}$ -FCP  $G^\times$  is the  $\mathcal{I}$ -FCP  $\text{GL}_1^\bullet A$  of units of  $A$ . Since  $Y$  is  $qf$ -fibrant, the derived fiber  $F_b Y$  over a point  $b \in B$  is represented by the fiber  $Y_b$ . Choose a  $q$ -equivalence of  $G$ -modules  $Y_b \simeq G$ . We then have an isomorphism of  $\pi_0 G$ -modules  $\pi_0 Y_b \cong \pi_0 G$ . Define  $\pi_0 Y_b^\times$  to be the subset of  $\pi_0 Y_b$  corresponding to  $\pi_0 G^\times$  under this isomorphism. Although the isomorphism  $\pi_0 Y_b^\times \cong \pi_0 G^\times$  of  $\pi_0 G^\times$ -modules depends on the choice of  $q$ -equivalence  $Y_b \simeq G$ , the subset  $\pi_0 Y_b^\times$  does not.

Let  $\pi_0^B Y^\times \subset \pi_0^B Y$  be the subspace consisting of the sets  $\pi_0 Y_b^\times$  in each fiber. Define the  $\mathcal{I}$ -space  $Y^\times$  over  $B$  to be the following pullback of  $\mathcal{I}$ -spaces:

$$\begin{array}{ccc} Y^\times & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \pi_0^B Y^\times & \longrightarrow & \pi_0^B Y \end{array}$$

**Proposition 7.4.** *Suppose that  $Y$  is a  $G$ -torsor over  $B$  that is  $qf$ -fibrant as a  $G$ -module over  $B$ .*

- (i)  $Y^\times$  is a  $G^\times$ -torsor over  $B$
- (ii)  $Y^\times$  is  $qf$ -fibrant as a  $G^\times$ -module over  $B$ .
- (iii) The construction  $Y \mapsto Y^\times$  is functorial and preserves  $q$ -equivalences between  $qf$ -fibrant  $G$ -torsors.

*Proof.* (i) In order to define a  $G^\times$ -module structure on  $Y^\times$ , we will show that the composite

$$\alpha^\times : G^\times \boxtimes Y^\times \longrightarrow G \boxtimes Y \xrightarrow{\alpha} Y$$

of the canonical inclusions followed by the action of  $G$  on  $Y$  factors through  $Y^\times$ . By the construction of  $Y^\times$ , it suffices to show that  $\alpha^\times$  factors through  $Y^\times$  on  $\pi_0$  of each fiber. To this end, consider the following diagram, in which the left vertical map is induced by the lax monoidal structure map  $l$  of the homotopy colimit functor  $(-)_h\mathcal{I}$ .

$$\begin{array}{ccc} \pi_0 G^\times \times \pi_0 Y_b^\times & \longrightarrow & \pi_0 Y_b^\times \\ \pi_0 l \downarrow & \nearrow & \downarrow \\ \pi_0 (G^\times \boxtimes Y_b^\times) & \xrightarrow{\alpha^\times} & \pi_0 Y_b \end{array}$$

The definition of the invertible components  $\pi_0 Y_b^\times$  implies that the action of  $\pi_0 G^\times$  on  $\pi_0 Y_b^\times$  lands in  $\pi_0 Y_b^\times$  as indicated by the commutative square. The map  $\pi_0 l$  is an isomorphism by Lemma 3.1, so there exists a diagonal lift and  $\alpha^\times$  factors through  $Y^\times$  as claimed. The associativity and unit conditions for the action follow from the associativity and unit conditions for  $\alpha$ .

(ii) At each level, the map  $Y^\times(V) \longrightarrow Y(V)$  is the inclusion of a set of connected components. It follows that the restriction of the structure map  $Y^\times \longrightarrow B$  is a  $qf$ -fibration of  $\mathcal{I}$ -spaces.

(iii) Suppose that  $f : X \longrightarrow Y$  is a  $q$ -equivalence of  $qf$ -fibrant  $G$ -torsors over  $B$ . Then  $f$  is a level-wise  $q$ -equivalence. Since  $X^\times$  and  $Y^\times$  are both  $qf$ -fibrant  $\mathcal{I}$ -spaces over  $B$ , it follows that the induced map  $f^\times : X^\times \longrightarrow Y^\times$  is also a level-wise  $q$ -equivalence, hence a  $q$ -equivalence.  $\square$

**Definition 7.5.** Let  $N$  be an  $s$ -fibrant  $R$ -bundle with fiber  $M$ . Since  $M$  and  $M^\circ$  are both  $s$ -cofibrant  $R$ -modules, the  $\text{End}_R M$ -torsors  $\Omega_B^\bullet F^R(M, N)$  and  $\Omega_B^\bullet F^R(M^\circ, N)$  are  $qf$ -fibrant as  $\text{End}_R M$ -modules. We let

$$E^R(M, N) = (\Omega_B^\bullet F_B^R(M, N))^\times \quad \text{and} \quad E^R(M^\circ, N) = (\Omega_B^\bullet F_B^R(M^\circ, N))^\times$$

be the associated  $\text{Aut}_R M$ -torsors. By neglect of structure,  $E^R(M, N)$  and  $E^R(M^\circ, N)$  are both  $\text{Aut}_R^c M$ -torsors as well. When  $N$  is an  $R$ -bundle with fiber  $M$  (but not necessarily  $s$ -fibrant), we define the  $\text{Aut}_R^c M$ -torsor associated to  $N$  to be  $E^R(M^\circ, N')$ , where  $N'$  is an  $s$ -fibrant approximation of the parametrized  $R$ -module  $N$ .

We now describe the inverse construction.

**Definition 7.6.** Let  $Y$  be an  $\text{Aut}_R^c M$ -torsor over  $B$ . The fiberwise suspension spectrum  $\Sigma_B^\bullet Y$  is a  $\Sigma_+^\bullet \text{Aut}_R^c M$ -module spectrum over  $B$ , hence a  $\Sigma_+^\bullet \text{Aut}_R^c M$ -module spectrum by neglect of structure. We define the  $R$ -bundle associated to  $Y$  to be the  $R$ -module  $T(Y) = M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_B^\bullet Y$  over  $B$ .

In order to prove that this construction actually gives an  $R$ -bundle with fiber  $M$ , we will need to prove an auxiliary result on commutation of right and left derived functors, which we postpone to the next section.

**Remark 7.7.** In the case  $M = R$ , our definition recovers the construction of Thom spectra from [3, 4, 16]. Given a map of spaces  $f: B \rightarrow BGL_1 R$ , the classification of principal  $GL_1 R$ -fibrations gives a principal  $GL_1 R$ -fibration  $Y_f$  over  $B$ . Applying the functor  $T$  then gives a rank one  $R$ -bundle over  $B$ . The Thom spectrum associated to the map  $f$  is the (non-parametrized)  $R$ -module spectrum

$$Mf = r_! TY_f \cong R^\circ \wedge_{\Sigma_+^\bullet GL_1^c R} \Sigma_+^\bullet Y_f,$$

where  $r_!: \mathcal{S}_B \rightarrow \mathcal{S}$  is left adjoint to the pullback functor  $r^*: \mathcal{S} \rightarrow \mathcal{S}_B$ .

## 8. THE CLASSIFICATION OF $R$ -BUNDLES

In this section, we show that the derived functors of  $T = M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_B^\bullet (-)$  and  $E^R(M^\circ, -)$  induce an equivalence between equivalence classes of  $\text{Aut}_R^c M$ -torsors and  $R$ -bundles with fiber  $M$ . Along with the classification theorem for principal  $\text{Aut}_R^c M$ -fibrations and the equivalence between principal  $\text{Aut}_R^c M$ -fibrations and  $\text{Aut}_R^c M$ -torsors, this will complete the proof of Theorem 1.2.

The fiber functor  $(-)_b = i_b^*$  is a left adjoint, but is not left Quillen for either the stable model structure on parametrized spectra or the  $qf$ -model structure on parametrized  $\mathcal{I}$ -spaces. Instead,  $i_b^*$  is a right Quillen functor. On the other hand,  $M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} (-)$  is a left Quillen functor (Proposition 6.9). There is a natural isomorphism of functors

$$(M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_B^\bullet Y)_b \cong M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_+^\bullet Y_b$$

at the point-set level, but we are not guaranteed an isomorphism of derived functors after passage to homotopy categories because we are composing left and right derived functors.

When  $B = *$ , the fact that  $M^\circ$  is  $s$ -cofibrant as a  $\Sigma_+^\bullet \text{Aut}_R^c M$ -module implies that the functor  $M^\circ \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} (-)$  preserves stable equivalences [14, 12.7]. Along with Lemma 6.5, this shows that the functor  $T$  takes  $q$ -equivalences to stable equivalences when the base is a point. This fact will allow us to deduce the desired commutation of derived functors. The proof of the next result is inspired by Shulman's examples in §9 of [22].

**Lemma 8.1.** *Let  $f: * \rightarrow B$  be the inclusion of a point. Then there is a natural isomorphism of derived functors  $\mathbf{R}f^* \mathbf{L}T \cong \mathbf{L}T \mathbf{R}f^*$ .*

*Proof.* Suppose that  $X$  is a  $qf$ -bifibrant  $\text{Aut}_R^c M$ -module over  $B$ , and consider the following natural transformation of  $R$ -modules

$$(8.1) \quad TQf^*X \longrightarrow Tf^*X \xrightarrow{\cong} f^*TX \longrightarrow f^*RTX,$$

where the first and third maps are induced by  $qf$ -cofibrant approximation and  $s$ -fibrant approximation, respectively. Since  $T$  preserves all weak equivalences when the base is a point, the first map is a stable equivalence. The second map is the canonical isomorphism. It remains to show that  $f^*$  preserves the stable equivalence  $TX \longrightarrow RTX$ .

Factor  $f$  as a  $q$ -equivalence followed by a  $q$ -fibration, and consider the two cases separately. In the first case, the Quillen adjunction  $(f_!, f^*)$  between module categories over  $*$  and  $B$  is a Quillen equivalence (the case of  $R = S$  is [18, 12.6.7] and the general case follows since stable equivalences and  $s$ -fibrations of  $R$ -modules are detected by the forgetful functor to parametrized spectra). It follows that the natural transformation of derived functors

$$\mathbf{L}T\mathbf{R}f^* \xrightarrow{\eta} \mathbf{R}f^*\mathbf{L}f_!\mathbf{L}T\mathbf{R}f^* \cong \mathbf{R}f^*\mathbf{L}T\mathbf{L}f_!\mathbf{R}f^* \xrightarrow{\epsilon} \mathbf{R}f^*\mathbf{L}T$$

is an isomorphism. As discussed in [22, §7], this isomorphism of derived functors is represented by the composite (8.1). In particular,  $f^*TX \longrightarrow f^*RTX$  is a stable equivalence in this case, since the map  $f$  is still the inclusion of a point.

When  $f$  is a  $q$ -fibration, we instead consider a level-wise  $qf$ -fibrant approximation  $TX \longrightarrow R_lTX$ . There is a stable equivalence  $R_lTX \longrightarrow RTX$  under  $TX$  [18, 12.6.1] and the induced map  $f^*R_lTX \longrightarrow f^*RTX$  is a stable equivalence because  $f^*$  preserves stable equivalences between level-wise  $qf$ -fibrant spectra. Pull-back along  $q$ -fibrations preserves weak homotopy equivalences of topological spaces, so  $f^*TX \longrightarrow f^*R_lTX$  is a level-wise  $q$ -equivalence, hence a stable equivalence. Therefore  $f^*TX \longrightarrow f^*RTX$  is also a stable equivalence.  $\square$

We will now use bold-face letters to denote derived functors, so that  $\mathbf{T}$  is the left derived functor of  $T$  and  $\mathbf{E}$  is the right derived functor of  $E = E^R(M^\circ, -)$ . We let  $\mathbf{F}_b = \mathbf{R}i_b^*$  denote the right derived fiber functor on spectra over  $B$  and  $\mathcal{I}$ -spaces over  $B$ .

**Proposition 8.2.** *There are natural isomorphisms of derived functors  $\mathbf{F}_b\mathbf{T} \cong \mathbf{T}\mathbf{F}_b$  and  $\mathbf{F}_b\mathbf{E} \cong \mathbf{E}\mathbf{F}_b$ .*

*Proof.* The first isomorphism is Lemma 8.1. The second isomorphism follows from the natural isomorphism  $E^R(M^\circ, N)_b \cong E^R(M^\circ, N_b)$  in the category of  $\text{Aut}_R^c M$ -modules and the fact that  $i_b^*$  and  $E$  both preserve fibrant objects.  $\square$

The cofibrant approximation map  $M^\circ \longrightarrow M$  is a stable equivalence of  $R$ -modules, so the derived functors  $\mathbf{E} = \mathbf{E}^R(M^\circ, -)$  and  $\mathbf{E}^R(M, -)$  are canonically isomorphic. To ease the exposition in the next proof we will identify these two functors.

**Theorem 8.3.** *The pair of functors  $(\mathbf{T}, \mathbf{E})$  gives an equivalence between the set of  $q$ -equivalence classes of  $\text{Aut}_R^c M$ -torsors over  $B$  and the set of stable equivalence classes of  $R$ -bundles with fiber  $M$  over  $B$ .*

Along with Theorem 1.3 and the equivalence between principal  $\text{Aut}_R^c M$ -fibrations and  $\text{Aut}_R^c M$ -torsors in Proposition 5.6, this theorem completes the proof of Theorem 1.2.

*Proof.* It is a consequence of Proposition 8.2 that the derived functor  $\mathbf{T}$  takes  $\text{Aut}_R^c M$ -torsors to  $R$ -bundles with fiber  $M$  and that  $\mathbf{E}$  takes  $R$ -bundles with fiber  $M$  to  $\text{Aut}_R^c M$ -torsors.

Suppose that  $Y$  is an  $\text{Aut}_R^c M$ -torsor over  $B$ . We will construct a natural transformation of derived functors  $\zeta: Y \rightarrow \mathbf{E}\mathbf{T}Y$  by showing that the composite of the units of the adjunctions  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  and  $(\mathbf{T}, \mathbf{F}^R(M, -))$  factors through  $\mathbf{E}^R(M, \mathbf{T}Y)$  as indicated in the following diagram.

$$(8.2) \quad \begin{array}{ccc} Y & \xrightarrow{\eta} & \Omega_B^\bullet \Sigma_B^\bullet Y \xrightarrow{\eta} \Omega_B^\bullet \mathbf{F}^R(M, \mathbf{T}Y) \\ & \searrow \zeta & \uparrow \iota \\ & & \mathbf{E}^R(M, \mathbf{T}Y) \end{array}$$

By the construction of  $\mathbf{E}^R(M, -)$ , it suffices to show that the factorization exists on  $\pi_0$  along each fiber after taking a fibrant approximation, and for this it suffices to show that the factorization exists in the derived category of  $\text{Aut}_R^c M$ -modules after applying the derived fiber functor  $\mathbf{F}_b$ . Apply the derived fiber functor  $\mathbf{F}_b$  to diagram (8.2) and commute  $\mathbf{F}_b$  past the constituent functors to the input variable  $Y$ . Now fix an isomorphism in the derived category  $\mathbf{F}_b Y \cong \text{Aut}_R^c M$  and consider the isomorphic diagram with  $\mathbf{F}_b Y$  replaced by  $\text{Aut}_R^c M$ . The composite of the two instances of  $\eta$  in this new diagram is the left vertical composite in the following commutative diagram.

$$\begin{array}{ccc} \text{Aut}_R^c M & \xrightarrow{\quad} & \text{End}_R M \\ \downarrow & & \downarrow \\ \Omega^\bullet \Sigma_+^\bullet \text{Aut}_R^c M & \xrightarrow{\quad} & \Omega^\bullet \Sigma_+^\bullet \text{End}_R M \\ \downarrow & & \downarrow \\ \Omega^\bullet \mathbf{F}^R(M, M \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_+^\bullet \text{Aut}_R^c M) & \xrightarrow{\quad} & \Omega^\bullet \mathbf{F}^R(M, M \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_+^\bullet \text{End}_R M) \\ & \searrow \cong & \downarrow \\ & & \Omega^\bullet \mathbf{F}^R(M, M) \end{array}$$

Here the horizontal maps are induced by the composite  $\text{Aut}_R^c M \rightarrow \text{Aut}_R M \rightarrow \text{End}_R M$  of the cofibrant approximation map and the canonical inclusion. The diagonal map is induced by the action map

$$M \wedge_{\Sigma_+^\bullet \text{Aut}_R^c M} \Sigma_+^\bullet \text{Aut}_R^c M \rightarrow M$$

for the right  $\Sigma_+^\bullet \text{Aut}_R^c M$ -module structure on  $M$  and it is an isomorphism as indicated. Since  $M$  is bifibrant, we may choose to represent the derived functor  $\Omega^\bullet \mathbf{F}^R(M, M)$  in the homotopy category by  $\text{End}_R M$ . A diagram chase involving the triangle identities for the adjunctions  $(\Sigma_+^\bullet, \Omega^\bullet)$  and  $(\mathbf{T}, \mathbf{F}^R(M, -))$  shows that the right vertical composite is then the identity map. It follows that the left vertical composite factors through  $\text{Aut}_R M = \mathbf{E}^R(M, M)$  via the cofibrant approximation map. This verifies the requested factorization in diagram (8.2), and so we have constructed the natural transformation  $\zeta: Y \rightarrow \mathbf{E}\mathbf{T}Y$ .

As a consequence of the preceding argument, we see that  $\mathbf{F}_b\zeta$  is equivalent to the cofibrant approximation map  $\mathrm{Aut}_R^c M \rightarrow \mathrm{Aut}_R M$ . It follows that  $\zeta$  is a fiberwise equivalence, and thus induces a natural isomorphism of derived functors.

Now let  $N$  be an  $R$ -bundle with fiber  $M$ . Define  $\xi: \mathbf{T}EN \rightarrow N$  to be the composite

$$\mathbf{T}\Sigma_B^\bullet \mathbf{E}^R(M, N) \xrightarrow{\iota} \mathbf{T}\Sigma_B^\bullet \Omega_B^\bullet \mathbf{F}^R(M, N) \xrightarrow{\epsilon} \mathbf{T}\mathbf{F}^R(M, N) \xrightarrow{\epsilon} N$$

of the map induced by the inclusion  $\iota: \mathbf{E}^R(M, N) \rightarrow \Omega_B^\bullet \mathbf{F}^R(M, N)$  followed by the counits for the adjunctions  $(\Sigma_B^\bullet, \Omega_B^\bullet)$  and  $(\mathbf{T}, \mathbf{F}^R(M, -))$ . After applying the derived fiber functor  $\mathbf{F}_b$ , commuting it through to the variable  $N$ , and using a chosen equivalence  $\mathbf{F}_b N \simeq M$ , an argument similar to that just given for  $\zeta$  proves that  $\mathbf{F}_b\xi$  is a fiberwise equivalence. Hence  $\xi$  also induces a natural isomorphism of derived functors.  $\square$

## 9. LIFTED $R$ -BUNDLES AND ALGEBRAIC $K$ -THEORY

In this section we will prove Theorem 1.1. Having used diagram spaces to prove the classification theorem for  $R$ -bundles with fiber  $M$ , we now return to the category of spaces for the following discussion. The arguments are adapted from [5, 10].

Let  $X$  be a finite CW complex and let  $R$  be a connective cofibrant orthogonal ring spectrum. Let

$$\mathrm{GL}_n R = \mathbb{Q}_* \mathrm{Aut}_R^c(R^{\vee n})$$

be the grouplike  $A_\infty$  space associated to a  $q$ -cofibrant approximation of the grouplike  $\mathcal{I}$ -FCP  $\mathrm{Aut}_R(R^{\vee n})$ . By Theorem 1.2, the classifying space  $B\mathrm{GL}_n R$  classifies stable equivalence classes of  $R$ -bundles with fiber  $R^{\vee n}$ . Let  $B\mathrm{GL}_\infty R = \mathrm{colim}_n B\mathrm{GL}_n R$ . Recall the following description of the zeroeth space of the algebraic  $K$ -theory spectrum of  $R$ :

$$\Omega^\infty K(R) \simeq K_0 R \times B\mathrm{GL}_\infty R^+$$

The group  $K_0 R = K_0^f \pi_0 R$  is the algebraic  $K$ -theory of free  $\pi_0 R$ -modules, and the plus denotes Quillen's plus construction with respect to the commutator subgroup of  $\pi_1 B\mathrm{GL}_\infty R$ . Since the plus construction changes the homotopy type in general, we will need to work with lifted bundles, in the following sense.

**Definition 9.1.** A lifted  $R$ -bundle over  $X$  is the data of:

- (i) An  $H_*$ -acyclic fibration  $p: Y \rightarrow X$  of CW complexes, by which we mean a  $q$ -fibration with  $\tilde{H}_*(\mathrm{fiber}(p); \mathbf{Z}) = 0$ .
- (ii) An  $R$ -bundle  $E$  over  $Y$ .

We say that a lifted  $R$ -bundle  $(E, Y, p)$  over  $X$  is free if every fiber of  $E$  admits a stable equivalence of  $R$ -modules  $E_y \simeq R^{\vee n}$  for some  $n$ .

Define a relation on lifted  $R$ -bundles over  $X$  by declaring  $(E, Y, p) \sim (E', Y', p')$  if there exists a map  $f: Y \rightarrow Y'$  over  $X$  such that the induced map of  $R$ -modules  $E \rightarrow f^* E'$  over  $Y$  is a stable equivalence. This does *not* define an equivalence relation in general, so we will work with the equivalence relation on lifted  $R$ -bundles over  $E$  generated by  $\sim$ .

We assume from now on that  $X$  is a finite CW complex. Let  $\Phi_R(X)$  be the set of equivalence classes of lifted free  $R$ -bundles over  $X$ . The set  $\Phi_R(X)$  is an abelian

monoid, where the sum  $(E_1, Y_1) \oplus (E_2, Y_2)$  of two lifted  $R$ -bundles over  $X$  is the lifted  $R$ -bundle

$$(g_1^* E_1 \vee_Z g_2^* E_2, Z), \quad \text{where } Z \text{ is the pullback}$$

$$\begin{array}{ccc} Z & \xrightarrow{g_2} & Y_2 \\ g_1 \downarrow & & \downarrow \\ Y_1 & \longrightarrow & X \end{array}$$

The zero of  $\Phi_R(X)$  is the trivial  $R$ -bundle  $(*_X, X)$  over  $X$ . Let  $\overline{K}_R(X)$  be the Grothendieck group of the monoid  $\Phi_R(X)$ .

We say that a lifted  $R$ -bundle is virtually trivial if there exists a space  $T$  such that  $\tilde{H}_*(T; \mathbf{Z}) = 0$  and a map  $f: Y \rightarrow T$  (not necessarily over  $X$ ) along with an  $R$ -bundle  $(E', T)$  over  $T$  and a stable equivalence of  $R$ -bundles  $E \simeq f^* E'$ .

**Lemma 9.2.** *Let  $(E_1, Y_1)$  be a lifted free  $R$ -bundle over  $X$ . Then there exists a lifted free  $R$ -bundle  $(E_2, Y_2)$  over  $X$  such that  $(E_1, Y_1) \oplus (E_2, Y_2)$  is virtually trivial.*

*Proof.* Let  $f_1: Y_1 \rightarrow BGL_n R$  be a classifying map for  $E_1$ . Let  $P$  be the homotopy fiber of the  $H_*$ -acyclic fibration  $Y_1 \rightarrow X$ . By [9, 1.3], the kernel of  $\pi_1 Y_1 \rightarrow \pi_1 X$  is the perfect normal subgroup  $\text{im}(\pi_1 P \rightarrow \pi_1 Y_1)$ . This is annihilated by the following map to the plus construction:

$$\pi_1 P \rightarrow \pi_1 Y_1 \xrightarrow{f_1} \pi_1 BGL_n R \rightarrow \pi_1 BGL_n R^+.$$

By [9, 3.1],  $f_1$  descends to a map  $g_1: X \rightarrow BGL_n R^+$ . Use the grouplike  $H$ -space structure on  $BGL_\infty R^+$  to find  $g_2: X \rightarrow BGL_m R^+$  such that  $g_1 \oplus g_2: X \rightarrow BGL_{m+n} R^+$  is nullhomotopic. Define  $Y_2$  as the following pullback:

$$\begin{array}{ccc} Y_2 & \xrightarrow{f_2} & BGL_m R \\ \downarrow & & \downarrow \\ X & \xrightarrow{g_2} & BGL_m R^+ \end{array}$$

We choose a model for the plus construction such that the right vertical map (and thus the left vertical map) is a  $q$ -fibration of CW complexes. Let  $E_2$  be the free  $R$ -bundle over  $Y_2$  classified by the map  $f_2$ . The sum  $(E_1, Y_1) \oplus (E_2, Y_2)$  is a lifted  $R$ -bundle over the pullback  $Y = Y_1 \times_X Y_2$  that is classified by a lift  $f: Y \rightarrow BGL_{m+n} R$  of  $g_1 \oplus g_2$ . Thus  $f$  is nullhomotopic, so it factors through the  $H_*$ -acyclic fiber of  $BGL_{m+n} R \rightarrow BGL_{m+n} R^+$ , proving that  $(E_1, Y_1) \oplus (E_2, Y_2)$  is virtually trivial.  $\square$

**Lemma 9.3.** *Suppose that  $(E, Y)$  is a virtually trivial lifted  $R$ -bundle over  $X$ . Then there exists a lifted  $R$ -bundle  $(r^* M, Y')$  over  $X$  that is equivalent to  $(E, Y)$  as a lifted  $R$ -bundle:  $[(E, Y)] = [(r^* M, Y')] \text{ in } \Phi_R(X)$ . If  $E$  is a free  $R$ -bundle, then  $M = R^{\vee n}$  for some  $n$ .*

*Proof.* We are given  $f: Y \rightarrow T$  where  $\tilde{H}_*(T) = 0$  and a stable equivalence  $E \simeq f^* E'$  where  $E'$  is an  $R$ -bundle over  $T$ . Choose a point  $p: * \rightarrow T$ . Consider the

following commutative diagram:

$$\begin{array}{ccccc}
 & & Y & & \\
 & g \swarrow & \downarrow \tau & \searrow f & \\
 X & \xleftarrow{\pi_2} & T \times X & \xrightarrow{\pi_1} & T \\
 & \nwarrow \text{id} & \uparrow \chi & & \uparrow p \\
 & & X & \xrightarrow{r} & *
 \end{array}$$

where  $\tau(y) = (f(y), g(y))$ ,  $\chi(x) = (p, x)$ . The maps  $g, \pi_2$  and  $\text{id}$  are all  $H_*$ -acyclic fibrations. Form the  $R$ -bundle  $\pi_1^* E'$  over  $T \times X$ . Then we have a stable equivalence of  $R$ -bundles over  $Y$ :  $\tau^* \pi_1^* E' = f^* E' \simeq E$ . On the other hand  $\chi^* \pi_1^* E' = r^* p^* E'$ , which is a trivial bundle over  $X$  with fiber  $M = p^* E'$ , since  $p \circ r$  factors through a point. The two triangles on the left show that  $(E, Y) \sim (\pi_1^* E', T \times X)$  and  $(\pi_1^* E', T \times X) \sim (r^* M, X)$ .  $\square$

Let  $\psi: \overline{K}_R(X) \rightarrow [X, K_0(R)]$  be the extension of the map of monoids  $\Phi_R(X) \rightarrow [X, K_0(R)]$  that takes a lifted free  $R$ -bundle to the class of the fiber in  $K_0(R) = K_0^f(\pi_0 R)$  over each component. Here  $K_0(R)$  is a discrete space. There is a natural splitting

$$\overline{K}_R(X) \cong \ker \psi \oplus [X, K_0(R)].$$

Let  $\Phi_R^n(X)$  be the set of equivalence classes of lifted  $R$ -bundles of rank  $n$ .

**Proposition 9.4.** *There is a natural isomorphism*

$$\ker \psi \cong \text{colim}_n \Phi_R^n(X).$$

*Proof.* Suppose that  $[E] - [F]$  is a formal difference of lifted free  $R$ -bundles in  $\ker \psi$ . We associate to  $[E] - [F]$  the element  $[E \oplus F'] \in \text{colim}_n \Phi_R^n(X)$  where  $F'$  is a lifted free  $R$ -bundle such that  $F \oplus F'$  is virtually trivial (Lemma 9.2). Conversely, to a class  $[E] \in \Phi_R^n(X)$  we associate the formal difference  $[E] - [T_n] \in \ker \psi$ , where  $T_n = r^* R^{\vee n}$  is the trivial bundle of rank  $n$ .  $\square$

**Proposition 9.5.** *There is a natural isomorphism*

$$\text{colim}_n \Phi_R^n(X) \cong [X, \text{BGL}_\infty(R)^+].$$

*Proof.* Given the class of a lifted free  $R$ -bundle  $(E, Y)$  over  $X$  in  $\text{colim}_n \Phi_R^n(X)$ , the arguments of Lemma 9.2 show that the classifying map  $f$  of  $E$  extends to a map  $g$  from  $X$  to the plus construction:

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & \text{BGL}_n R \\
 p \downarrow & & \downarrow \\
 X & \xrightarrow{g} & \text{BGL}_n R^+
 \end{array}$$

Conversely, given a classifying map  $g$  define  $Y$  as the pullback displayed in the same diagram. Then  $p$  is an  $H_*$ -acyclic fibration and  $f$  classifies a lifted free  $R$ -bundle  $(E, Y)$  over  $X$ .  $\square$

All together, we have proved:

$$\overline{K}_R(X) \cong [X, K_0(R)] \oplus [X, \text{BGL}_\infty(R)^+] \cong [X, \Omega^\infty K(R)].$$

This completes the proof of Theorem 1.1.

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